Line Integrals

P. Sam Johnson

National Institute of Technology Karnataka (NITK)
Surathkal, Mangalore, India

Overview

We discuss integration in vector fields in the lecture.

You must be knowing the definite integral of a function over a finite closed interval $[a, b]$ on the x-axis.

Definite integrals are used to find the mass of a thin straight rod, or the work done by a variable force directed along the x-axis.

Bucket and Rope

Leaky Bucket

Work Done by a Variable Force along the x-axis

A leaky 5 N bucket is lifted from the ground into the air by pulling in 20 meter of rope at a constant speed. The rope weighs 0.08 N/m. The bucket starts with 16 N of water and leaks at a constant rate. It finishes draining just as it reaches the top. How much work was done ?

- lifting the water alone ;
- lifting the water and bucket together ;
- lifting the water, bucket and rope?

Now we would like to calculate the masses of thin rods or wires lying along a curve in the plane or space, or to find the work done by a variable force acting along such a curve.

For these calculations we need a more general notion of a "line" integral than integrating over a line segment on the x-axis.

Instead we need to integrate over a curve C in the plane or in space. These more general integrals are called **line integrals**, although "curve" (or "path") integrals might be more descriptive.

We make our definitions for space curves, remembering that curves in the xy-plane are just a special case with z-coordinate identically zero.

Vector Functions

When a particle moves through space during a time interval I, we think of the particle's coordinates as functions defined on I:

$$x = f(t), \quad y = g(t), \quad z = h(t), \quad t \in I.$$

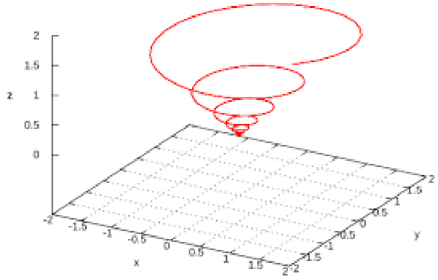
The points

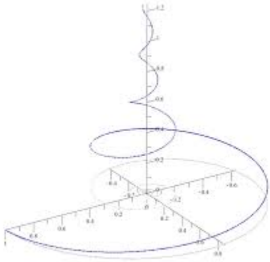
$$(x, y, z) = (f(t), g(t), h(t)), \ t \in I$$

made up the **curve** in space that we call the **particle's path**.

The equations and interval in the above equations **parameterize** the curve.

Space Curves





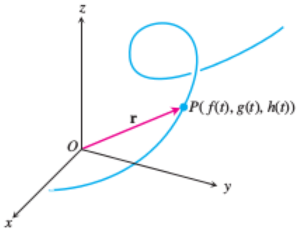
Vector Functions

A curve in space can also be represented in vector form. The vector

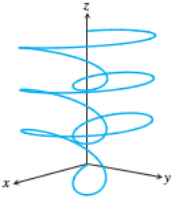
$$\boldsymbol{r}(t) = \boldsymbol{OP} = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$$

is the particle's **position vector**.

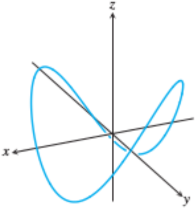
r(t) denotes the vector from the origin to the particle's **position** $P(f(t), g(t), h(t))$ at time t.



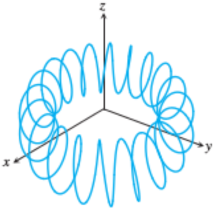
The following figures display several space curves.



$\mathbf{r}(t) = (\sin 3t)(\cos t)\mathbf{i} + (\sin 3t)(\sin t)\mathbf{j} + t\mathbf{k}$



$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (\sin 2t)\mathbf{k}$



$\mathbf{r}(t) = (4 + \sin 20t)(\cos t)\mathbf{i} + (4 + \sin 20t)(\sin t)\mathbf{j} + (\cos 20t)\mathbf{k}$

Vector Functions

The functions f, g, and h are the **component functions (components)** of the position vector. We think of the particle's path as the **curve traced by $\boldsymbol{r}$** during the time interval I.

The above definition defines $\boldsymbol{r}$ as a vector function of the real variable t on the interval I. More generally, a **vector function** or **vector-valued function** on a domain set D is a rule that assigns a vector in space to each element in D.

For now, the domains will be intervals of real numbers resulting in a space curve.

Vector Functions

Later, we see, the domains will be regions in the plane. Vector functions will then represent surfaces in space.

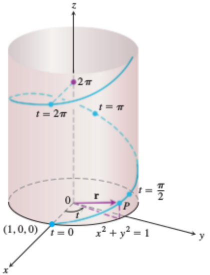
Vector functions on a domain in the plane or space also give rise to "vector fields," which are important to the study of the flow of a fluid, gravitational fields, and electromagnetic phenomena.

Vector Functions

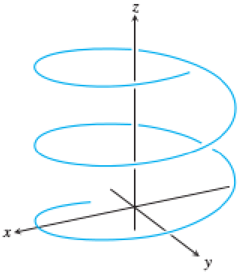
We refer to real-valued functions as **scalar functions** to distinguish them from vector functions. The components of $\boldsymbol{r}$ are scalar functions of t. When we define a vector-valued function by giving its component functions, **we assume that vector function's domain to be the common domain of the components.**

The following graph shows a helix by the vector function

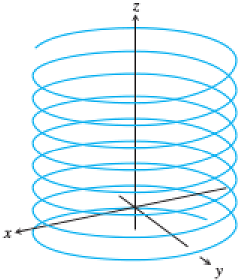
$$\boldsymbol{r}(t) = (\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j} + t\boldsymbol{k}.$$



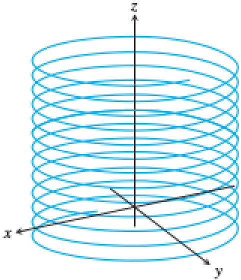
Other Helices



$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$



$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + 0.3t\mathbf{k}$



$\mathbf{r}(t) = (\cos 5t)\mathbf{i} + (\sin 5t)\mathbf{j} + t\mathbf{k}$

The way we define the limits of vector-valued functions is similar to the way we define limits of real-valued functions.

Definition 1.

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ *be a vector function and* $\mathbf{L}$ *a vector. We say that* $\mathbf{r}$ *has* limit L *as* t *approaches* t_0 *and write*

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L}$$

if, for every number $\varepsilon > 0$*, there exists a corresponding number* $\delta > 0$ *such that for all* t

$$0 < |t - t_0| < \delta \quad \Rightarrow |\mathbf{r}(t) - \mathbf{L}| < \varepsilon.$$

Limits and Continuity

If $\boldsymbol{L} = L_1\boldsymbol{i} + L_2\boldsymbol{j} + L_3\boldsymbol{k}$, then $\lim_{t\to t_0} \boldsymbol{r}(t) = \boldsymbol{L}$ precisely when

$$\lim_{t\to t_0} f(t) = L_1, \quad \lim_{t\to t_0} g(t) = L_2, \quad \lim_{t\to t_0} h(t) = L_3.$$

The equation

$$\lim_{t\to t_0} r(t) = \left(\lim_{t\to t_0} f(t)\right)\boldsymbol{i} + \left(\lim_{t\to t_0} g(t)\right)\boldsymbol{j} + \left(\lim_{t\to t_0} h(t)\right)\boldsymbol{k}$$

provides a practical way to calculate limits of vector functions.

Limits and Continuity

We define continuity for vector functions the same way we define continuity for scalar functions.

Definition 2.

A vector function $\boldsymbol{r}(t)$ *is* continuous at a point $t = t_0$ *in its domain if* $\lim_{t \to t_0} \boldsymbol{r}(t) = \boldsymbol{r}(t_0)$. *The function is* continuous *if it is continuous at every point in its domain.*

Hence $\boldsymbol{r}(t)$ is continuous at $t = t_0$ if and only if each component function is continuous there.

Example 3.

The function $\boldsymbol{g}(t) = (\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j} + \lfloor t \rfloor \boldsymbol{k}$ *is discontinuous at every integer, where the greatest integer function* $\lfloor t \rfloor$ *is discontinuous.*

Derivatives and Motion

Suppose that

$$\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$$

is the position vector of a particle moving along a curve in space and that f, g, and h are differentiable functions of t. Then the difference between the particle's positions at time t and time $t + \Delta t$ is

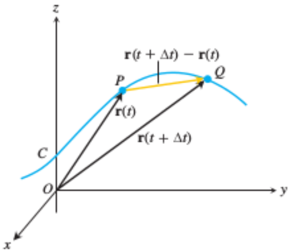
$$\Delta \boldsymbol{r} = \boldsymbol{r}(t + \Delta t) - \boldsymbol{r}(t).$$

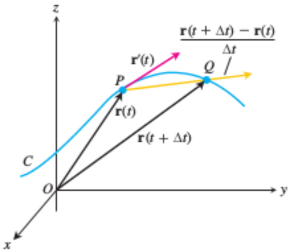
In terms of components,

$$\Delta \boldsymbol{r} = [f(t + \Delta t) - f(t)]\boldsymbol{i} + [g(t + \Delta t) - g(t)]\boldsymbol{j} + [h(t + \Delta t) - h(t)]\boldsymbol{k}.$$

As Δt approaches zero, three things seem to happen simultaneously.
First, Q approaches P along the curve.
Second, the secant line PQ seems to approach a limiting position tangent to the curve at P.
Third, the quotient $\Delta \boldsymbol{r}/\Delta t$ approaches the limit

$$\lim_{\Delta t \to 0} \Delta \boldsymbol{r}/\Delta t = \left[\frac{df}{dt}\right]\boldsymbol{i} + \left[\frac{dg}{dt}\right]\boldsymbol{j} + \left[\frac{dh}{dt}\right]\boldsymbol{k}.$$





Definition 4.

The vector function

$$\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$$

has a derivative at t *if* f, g, *and* h *have derivatives at* t.
The derivative is the vector function

$$\boldsymbol{r}'(t) = \frac{d\boldsymbol{r}}{dt} = \lim_{\Delta t \to 0} \frac{\boldsymbol{r}(t+\Delta t) - \boldsymbol{r}(t)}{\Delta t} = \left[\frac{df}{dt}\right]\boldsymbol{i} + \left[\frac{dg}{dt}\right]\boldsymbol{j} + \left[\frac{dh}{dt}\right]\boldsymbol{k}.$$

A vector function $\boldsymbol{r}$ is **differentiable** if it is differentiable at every point of its domain. The curve traced by $\boldsymbol{r}$ is **smooth** if $d\boldsymbol{r}/dt$ is continuous and never 0, that is, if f, g, and h have continuous first derivatives that are not simultaneously 0.

Derivatives and Motion

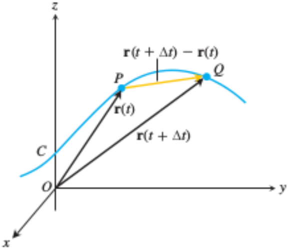
The geometric significance of the definition of derivative is shown in the figure. The points P and Q have position vectors $\boldsymbol{r}(t)$ and $\boldsymbol{r}(t+\Delta t) - \boldsymbol{r}(t)$, and the vector $\boldsymbol{PQ}$ is represented by

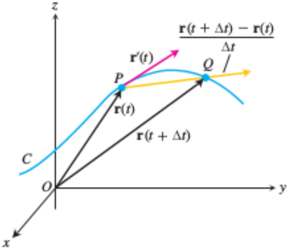
$$\boldsymbol{r}(t + \Delta t) - \boldsymbol{r}(t).$$

For $\Delta t > 0$, the scalar multiple

$$(1/\Delta t)(\boldsymbol{r}(t + \Delta t) - \boldsymbol{r}(t))$$

points in the same direction as the vector $\boldsymbol{PQ}$. As $\Delta t \to 0$, this vector approaches a vector defined to be the vector **tangent** to the curve at P.





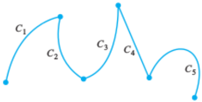
The **tangent line** to the curve at a point

$$(f(t_0), g(t_0), h(t_0))$$

is defined to be the line through the point parallel to $\boldsymbol{r}'(t_0)$. We require $d\boldsymbol{r}/dt \neq 0$ for a smooth curve to make sure the curve has a continuously turning tangent at each point.

On a smooth curve, there are no sharp corners or cusps.

A curve that is made up of a funite number of smooth curves pieced togther in a continuous fashion is called **piecewise smooth**.



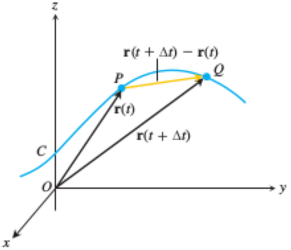
Derivatives and Motion

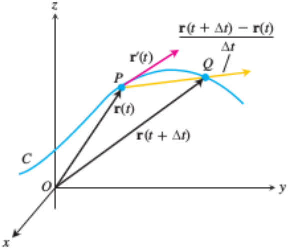
For Δt positive, $\Delta \boldsymbol{r}$ points forward, in the direction of the motion. The vector $\Delta \boldsymbol{r}/\Delta t$, having the same direction as $\Delta \boldsymbol{r}$ points forward too.

Had Δt been negative, $\Delta \boldsymbol{r}$ would have pointed backward, against the direction of the motion. The quotient vector $\Delta \boldsymbol{r}/\Delta t$, however, being a negative scalar multiple of $\Delta \boldsymbol{r}$ would once again have pointed forward.

Now matter how $\Delta \boldsymbol{r}$ points, $\Delta \boldsymbol{r}/\Delta t$ points forward and we expect the vector $d\boldsymbol{r}/dt = \lim_{\Delta \boldsymbol{r} \to 0} \Delta \boldsymbol{r}/\Delta t$, when different from 0, to do the same.

This means that the derivative $d\boldsymbol{r}/dt$ is just what we want for modeling a particle's velocity.





It points in the direction of motion and gives the rate of change of position with respect to time. For a smooth curve, the velocity is never zero; the particle does not stop or reverse direction.

Definition 5.

If $\boldsymbol{r}$ *is the position vector of a particle moving along a smooth curve in space, then*

$$\boldsymbol{v}(t) = \frac{d\boldsymbol{r}}{dt}$$

is the particle's velocity vector, *tangent to the curve.*
At any time t*, the direction of* $\boldsymbol{v}$ *is the* direction of motion, *the magnitute of* $\boldsymbol{v}$ *is the particle's* speed, *and the derivative* $\boldsymbol{a} = d\boldsymbol{v}/dt$*, when it exists, is the particle's* acceleration vector.

Derivatives and Motion

1. Velocity is the derivative of position : $\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt}$.
2. Speed is the magnitute of velocity : Speed $= |\boldsymbol{v}|$.
3. Acceleration is the derivative of velocity : $\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \frac{d^2\boldsymbol{r}}{dt^2}$.
4. The unit vector $\boldsymbol{v}/|\boldsymbol{v}|$ is the direction of motion at time t.
5. We can express the velocity of a moving particle as the product of its speed and direction :

$$\text{Velocity} = |\boldsymbol{v}|\left(\frac{\boldsymbol{v}}{|\boldsymbol{v}|}\right) = (\text{speed})(\text{direction}).$$

Because the derivatives of vector functions may be computed component by component, the rules for differentiating vector functions have the same form as the rules for differentiating scalar functions.

Differentiation Rules for Vector Functions

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be differentiable vector functions of t, $\boldsymbol{C}$ a constant vector, c any scalar, and f any differentiable scalar function.

1. Constant Function Rule : $\frac{d}{dt}\boldsymbol{C} = 0$
2. Scalar Multiple Rules : $\frac{d}{dt}[c\boldsymbol{u}(t)] = c\boldsymbol{u}'(t)$
$$\frac{d}{dt}[f(t)\boldsymbol{u}(t)] = f'(t)\boldsymbol{u}(t) + f(t)\boldsymbol{u}'(t)$$
3. Sum Rule : $\frac{d}{dt}[\boldsymbol{u}(t) + \boldsymbol{v}(t)] = \boldsymbol{u}'(t) + \boldsymbol{v}'(t)$
4. Difference Rule : $\frac{d}{dt}[\boldsymbol{u}(t) - \boldsymbol{v}(t)] = \boldsymbol{u}'(t) - \boldsymbol{v}'(t)$
5. Dot Product Rule : $\frac{d}{dt}[\boldsymbol{u}(t).\boldsymbol{v}(t)] = \boldsymbol{u}'(t).\boldsymbol{v}(t) + \boldsymbol{u}(t).\boldsymbol{v}'(t)$
6. Cross Product Rule : $\frac{d}{dt}[\boldsymbol{u}(t) \times \boldsymbol{v}(t)] = \boldsymbol{u}'(t) \times \boldsymbol{v}(t) + \boldsymbol{u}(t) \times \boldsymbol{v}'(t)$
7. Chain Rule : $\frac{d}{dt}[\boldsymbol{u}(f(t))] = f'(t)\boldsymbol{u}'(f(t))$.

Vector Functions of Constant Length

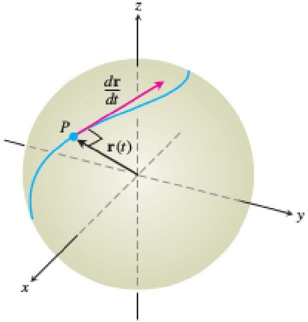
When we track a particle moving on a sphere centered at the origin, the position vector has a constant length equal to the radius of the sphere. The velocity vector $d\boldsymbol{r}/dt$, tangent to the path of motion, is tangent to the sphere and hence perpendicular to $\boldsymbol{r}$.

This is always the case for a differentiable vector function of constant length: The vector and its first derivative are orthogonal.

With the length constant, the change in the function is a change in direction only, and direction changes take place at right angles. We can also obtain this result by direct calculation : $\boldsymbol{r}(t).\boldsymbol{r}(t) = c^2$ gives, after differentiating both sides, $2\boldsymbol{r}'(t).\boldsymbol{r}(t) = 0$.

Vector Functions of Constant Length

If $\boldsymbol{r}$ is a differentiable vector function of t of constant length, then the vectors $\boldsymbol{r}'(t)$ and $\boldsymbol{r}(t)$ are orthogonal because their dot product is zero.



Integrals of Vector Functions

A differentiable vector function $\boldsymbol{R}(t)$ is an **antiderivative** of a vector function $\boldsymbol{r}(t)$ on an interval I if $d\boldsymbol{R}/dt = \boldsymbol{r}$ at each point of I. If $\boldsymbol{R}$ is an antiderivative of $\boldsymbol{r}$ on I, it can be shown, working one component at a time, that every antiderivative of $\boldsymbol{r}$ on I has the form $\boldsymbol{R} + \boldsymbol{C}$ for some constant vector $\boldsymbol{C}$.

The set of all antiderivatives of $\boldsymbol{r}$ on I is the **indefinite integral** of $\boldsymbol{r}$ on I.

The **indefinite integral** of $\boldsymbol{r}$ with respect to t is the set of all antiderivatives of r, denoted by $\displaystyle\int \boldsymbol{r}(t)\ dt$. If $\boldsymbol{R}$ is any antiderivative of $\boldsymbol{r}$, then

$$\int \boldsymbol{r}(t)\ dt = \boldsymbol{R}(t) + \boldsymbol{C}.$$

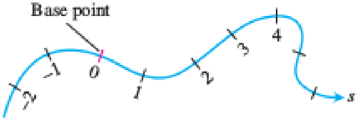
Definite integrals of vector functions are best defined in terms of components.

If the components of $\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$ are integrable over $[a, b]$, then so is $\boldsymbol{r}$, and the **definite integral** of r from a to b is

$$\int_a^b r(t)\ dt = \Big(\int_a^b f(t)\ dt\Big)\boldsymbol{i} + \Big(\int_a^b g(t)\ dt\Big)\boldsymbol{j} + \Big(\int_a^b h(t)\ dt\Big)\boldsymbol{k}.$$

Arc Length Along a Space Curve

One of the features of smooth space curves is that they have a measurable length. This enables us to locate points along these curves by giving their directed distance s along the curve from some **base point**, the way we locate points on coordinate axes by giving their directed distance from the origin.



Time is the natural parameter for describing a moving body's velocity and acceleration, but s is the natural parameter for studying a curve's shape.

Arc Length Along a Space Curve

To measure distance along a smooth curve in space, we add a z-term to the formula we use for curves in the plane.

The **length** of a smooth curve $\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}$, $a \leq t \leq b$, this is traced exactly once as t increases from $t = a$ to $t = b$, is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

The square root in the above equation is $|\boldsymbol{v}|$, the length of a velocity vector $d\boldsymbol{r}/dt$. This enables us to write the formula for length a shorter way.

Arc Length Formula :

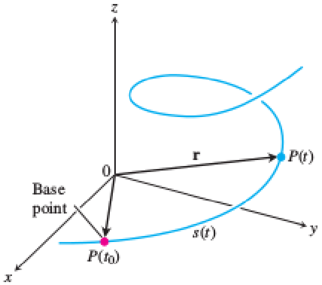
$$L = \int_a^b |\boldsymbol{v}|\, dt.$$

Arc Length Along a Space Curve

If we choose a base point $P(t_0)$ on a smooth curve C parameterized by t, each value of t determines a point $P(t) = (x(t), y(t), z(t))$ on C and a "directed distance"

$$s(t) = \int_{t_0}^{t} |v(\tau)| \, d\tau,$$

measured along C from the base point.



Arc Length Along a Space Curve

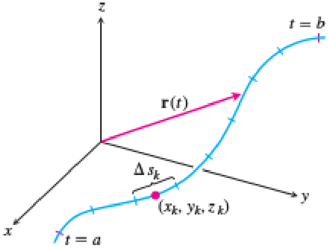
If $t > t_0$, $s(t)$ is the distance from $P(t_0)$ to $P(t)$. If $t < t_0$, $s(t)$ is the negative of the distance. Each value of s determines a point on C and this parameterizes C with respect to s. We call s an **arc length parameter** for the curve. The parameter's value increases in the direction of increasing t. **The arc length parameter is particularly effective for investigating the turning and twisting nature of a space curve.**

Since the curve is smooth, the Fundamental Theorem of Calculus tells us that s is a differentiable function of t with derivative $\frac{ds}{dt} = |\boldsymbol{v}(t)|$.

The **unit tangent vector** of a smooth vector $\boldsymbol{r}(t)$ is

$$T = \frac{d\boldsymbol{r}}{ds} = \frac{d\boldsymbol{r}/dt}{ds/dt} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|}.$$

Suppose that $f(x, y, z)$ is a real-valued function we wish to integrate over the curve $\boldsymbol{r}(t) = g(t)\boldsymbol{i} + h(t)\boldsymbol{j} + k(t)\boldsymbol{k}$, $a \leq t \leq b$, lying within the domain of f. The values of f along the curve are given by the composite function $f(g(t), h(t), k(t))$. We are going to integrate this composite with respect to arc length from $t = a$ to $t = b$. To begin, we first partition the curve into a finite number n of subarcs.



The typical subarc has length Δs_k. In each subarc we choose a point (x_k, y_k, z_k) and form the sum

$$S_n = \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta s_k. \tag{1}$$

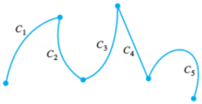
If f is continuous and the functions g, h, and k have continuous first derivatives, then these sums approach a limit as n increases and the lengths Δs_k approach zero. We call this limit the **line integral of f over the curve from a to b.** If the curve is denoted by a single letter, C for example, the notation for the integral is

$$\int_C f(x, y, z)\ ds \qquad \text{"The integral of } f \text{ over } C\text{".}$$

If $\boldsymbol{v} = d\boldsymbol{r}/dt$ is continuous and never 0, then the curve represented by $\boldsymbol{r}(t)$ for $a \leq t \leq b$, is called **smooth**.

On a smooth curve, there are no sharp corners or cusps.

A curve that is made up of a funite number of smooth curves pieced togther in a continuous fashion is called **piecewise smooth**.



If $\boldsymbol{r}(t)$ is smooth for $a \leq t \leq b$ and the function f is continuous on C, then the limit in Equation 1 can be shown to exist. We can then apply the Fundamental Theorem of Calculus to differentiate the arc length equation,

$$s(t) = \int_a^t |\boldsymbol{v}(\tau)| \, d\tau$$

to express ds in Equation 1 as $ds = |\boldsymbol{v}(t)| \, dt$.

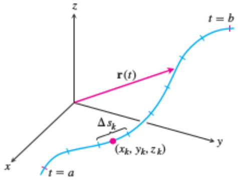
A theorem from advanced calculus says that we can then evaluate the integral of f over C as

$$\int_C f(x, y, z) \, ds = \int_a^b f(g(t), h(t), k(t)) |\boldsymbol{v}(t)| \, dt.$$

Notice that the integral on the right side of the last equation is just an ordinary (single) definite integral, where we are integrating with respect to the parameter t.

The formula evaluates the line integral on the left side correctly no matter what parameterization is used, as long as the parameterization is smooth.

Note that the parameter t defines a direction along the path. The starting point on C is the position $r(a)$ and movement along the path is in the direction of increasing t.

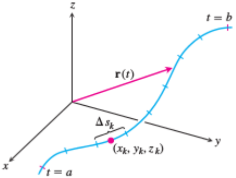


Additivity

Line integrals have the useful property that if a curve C is made by joining a finite number of curves $C_1, C_2, \ldots, C_n$ end to end, then the integral of a function over C is the sum of the integrals over the curves that make it up:

$$\int_C f\ ds = \int_{C_1} f\ ds + \int_{C_2} f\ ds + \cdots + \int_{C_n} f\ ds.$$

If f has the constant value 1, then the integral of f over C gives **the length of** C from $t = a$ to $t = b$.



How to Evaluate a Line Integral ?

To integrate a continuous function $f(x, y, z)$ over a curve C :

- Find a smooth parametrization of C,

$$\boldsymbol{r}(t) = g(t)\boldsymbol{i} + h(t)\boldsymbol{j} + k(t)\boldsymbol{k}, \quad a \leq t \leq b.$$

- Evaluate the integral as

$$\int\limits_C f(x, y, z)\ ds = \int\limits_a^b f(g(t), h(t), k(t))\ |\boldsymbol{v}(t)|\ dt.$$

Mass

We treat coil springs and wires like masses distributed along smooth curves in space. The distribution is described by a continuous density function $\delta(x, y, z)$ (mass per unit length).

The spring's or wire's mass is calculated with the following formula, which is applicable to thin rods.

Let $\delta(x, y, z)$ be the density at (x, y, z) mass per unit area.

Mass :

$$M = \int_C \delta(x, y, z)\ ds.$$

Moment Formulas for Coil Springs, Wires, and Thin Rods lying along a smooth curve C in space

First moments above the coordinate planes :

$$M_{yz} = \int_C x \; \delta \; ds, \quad M_{xz} = \int_C y \; \delta \; ds, \quad M_{xy} = \int_C z \; \delta \; ds.$$

Coordinates of the center of mass :

$$\overline{x} = M_{yz}/M, \quad \overline{y} = M_{xz}/M, \quad \overline{z} = M_{xy}/M.$$

Line Integrals in the Plane

There is an interesting geometric interpretation for line integrals in the plane. If C is a smooth curve in the xy-plane parametrized by

$$\mathrm{r}(t) = x(t)\mathrm{i} + y(t)\mathrm{j}, \quad a \leq t \leq b,$$

we generate a cylindrical surface by moving a straight line along C orthogonal to the plane, holding the line parallel to the z-axis.

If $z = f(x, y)$ is a nonnegative continuous function over a region in the plane containing the curve C, then the graph of f is a surface that lies above the plane.

The cylinder cuts through this surface, forming a curve on it that lies above the curve C and follows its winding nature. The part of the cylindrical surface that lies beneath the surface curve and above the xy-plane is like a "winding wall" or "fence" standing on the curve C and orthogonal to the plane.

At any point (x, y) along the curve, the height of the wall is $f(x, y)$.

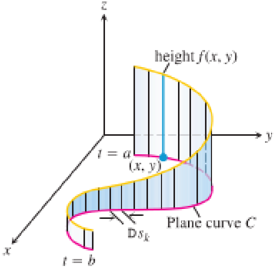


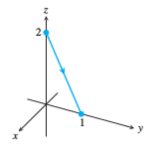
Figure shows that that the "top" of the wall is the curve lying on the surface $z = f(x, y)$.

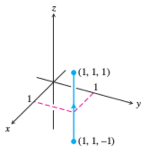
The line integral $\int_C f \; ds$ gives the area of the portion of the cylindrical surface or "wall" beneath $z = f(x, y) \geq 0$.

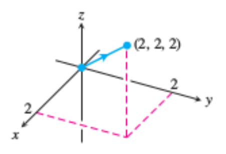
The line integral $\int_c f \; ds$ is the area of the wall shown in the figure.

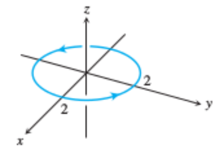
Exercise 6.

Write vector equations for the following graphs.

(a) 

(c) 

(b) 

(d) 

Solution for Exercise 6

(a) $r = t\mathbf{j} + (2 - 2t)\mathbf{k}$

(b) $r = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$

(c) $r = \mathbf{i} + \mathbf{j} + t\mathbf{k}$

(d) $r = (2\cos\ t)\mathbf{i} + (2\sin\ t)\mathbf{j}$

Exercise 7.

1. *Find the line integral of* $f(x, y, z) = x + y + z$ *over the straight-line segment from* $(1, 2, 3)$ *to* $(0, -1, 1)$.
2. *Integrate* $f(x, y, z) = -\sqrt{x^2 + z^2}$ *over the circle*

$$\mathrm{r}(t) = (a \cos t)\mathrm{j} + (a \sin t)\mathrm{k}, \qquad 0 \le t \le 2\pi.$$

3. *Integrate* $f(x, y) = x^2 - y$ *over the curve* $C : x^2 + y^2 = 4$ *in the first quadrant from* $(0, 2)$ *to* $(\sqrt{2}, \sqrt{2})$.
4. *Find the mass of a wire that lies along the curve*

$$\mathrm{r}(t) = (t^2 - 1)\mathrm{j} + 2t\mathrm{k}, \qquad 0 \le t \le 1$$

if the density is $\delta = (3/2)t$.

Solution for Exercise 7

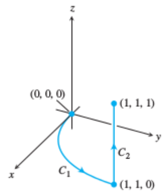
1. $\mathbf{r}(t) = (\mathbf{i}+2\mathbf{j}+3\mathbf{k}) + t(-\mathbf{i}-3\mathbf{j}-2\mathbf{k}) = (1-t)\mathbf{i} + (2-3t)\mathbf{j} + (3-2t)\mathbf{k}, 0 \le t \le 1 \Rightarrow \frac{dr}{dt} = -\mathbf{i}-3\mathbf{j}-2\mathbf{k} \Rightarrow |\frac{dr}{dt}| = \sqrt{1+9+4} = \sqrt{14}; x+y+z = (1-t)+(2-3t)+(3-2t) = 6-6t \Rightarrow \int_c f(x,y,z)ds = \int_0^1 (6-6t)\sqrt{14}dt = 6\sqrt{14}\left[t - \frac{t^2}{2}\right]_0^1 = (6\sqrt{14})(\frac{1}{2}) = 3\sqrt{14}$
2. $\mathbf{r}(t) = (a \ \cos \ t)\mathbf{j} + (a \ \sin \ t)\mathbf{k}, 0 \le t \le 2\pi \Rightarrow \frac{dr}{dt} = (-a \ \sin \ t)\mathbf{j} + (a\cos \ t)\mathbf{k} \Rightarrow |\frac{dr}{dt}| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = |a|; -\sqrt{x^2+z^2} = -\sqrt{0+a^2\sin^2 t} = \begin{cases} -|a|\sin \ t, 0 \le t \le \pi \\ |a|\sin t, \pi \le t \le 2\pi \end{cases} \Rightarrow \int_c f(x,y,z)ds = \int_0^\pi -|a|^2 \sin \ t \ dt + \int_\pi^{2\pi} |a|^2 \sin \ t \ dt = [a^2\cos \ t]_0^\pi - [a^2\cos \ t]_\pi^{2\pi} = [a^2(-1) - a^2] - [a^2 - a^2(-1)] = -4a^2$
3. $\mathbf{r}(t) = (2\sin \ t)\mathbf{i} + (2\cos t)\mathbf{j}, 0 \le t \le \frac{\pi}{4} \Rightarrow \frac{dr}{dt} = (2\cos \ t)\mathbf{i} + (-2\sin \ t)\mathbf{j} \Rightarrow |\frac{dr}{dt}| = 2; f(x,y) = f(2\sin \ t, 2\cos \ t) = 4\sin^2 t - 2\cos t \Rightarrow \int_c f \ ds = \int_0^{\pi/4}(4\sin^2 t - 2\cos \ t)(2)dt = [4t - 2 \ \sin \ 2t - 4\sin \ t]_0^{\pi/4} = \pi - 2(1+\sqrt{2})$
4. $\mathbf{r}(t) = (t^2-1)\mathbf{j} + 2t\mathbf{k}, 0 \le t \le 1 \Rightarrow \frac{dr}{dt} = 2t\mathbf{j} + 2\mathbf{k} \Rightarrow |\frac{dr}{dt}| = 2\sqrt{t^2+1}; M = \int_c \delta(x,y,z)ds = \int_0^1 \delta(t)(2\sqrt{t^2+1})dt = \int_0^1 (\frac{3}{2}t)(2\sqrt{t^2+1})dt = \left[(t^2+1)^{3/2}\right]_0^1 = 2^{3/2} - 1 = 2\sqrt{2} - 1$

Exercise 8.

Integrate $f(x, y, z) = x + \sqrt{y} - z^2$ *over the path from* $(0, 0, 0)$ *to* $(1, 1, 1)$ *given by*

$$C_1 : \mathrm{r}(t) = t\mathrm{i} + t^2\mathrm{j}, \quad 0 \le t \le 1.$$

$$C_2 : \mathrm{r}(t) = \mathrm{i} + \mathrm{j} + t\mathrm{k}, \quad 0 \le t \le 1.$$



How about integrating f *from* $(0, 0, 0)$ *to* $(1, 1, 1)$ *by a straight line? Note that the value of the line integral along a path joining two points can change if we change the path between them.*

$C_1 : \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, 0 \leq t \leq 1 \Rightarrow \frac{dr}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow |\frac{dr}{dt}| = \sqrt{1+4t^2}; x + \sqrt{y} - z^2 = t + \sqrt{t^2} - 0 = t + |t| = 2t$ since $t \geq 0 \Rightarrow \int_{C_1} f(x,y,z)ds = \int_0^1 2t\sqrt{1+4t^2}dt = \left[\frac{1}{6}(1+4t^2)^{3/2}\right]_0^1 = \frac{1}{6}(5)^{3/2} - \frac{1}{6} = \frac{1}{6}(5\sqrt{5}-1).$

$C_2 : \mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k}, 0 \leq t \leq 1 \Rightarrow \frac{dr}{dt} = \mathbf{k} \Rightarrow |\frac{dr}{dt}| = 1; x + \sqrt{y} - z^2 = 1 + \sqrt{1} - t^2 = 2 - t^2 \Rightarrow \int_{C_2} f(x,y,z)ds = \int_0^1 (2-t^2)(1)dt = [2t - \frac{1}{3}t^3]_0^1 = 2 - \frac{1}{3} = \frac{5}{3}.$

Therefore $\int_C f(x,y,z)ds = \int_{C_1} f(x,y,z)ds + \int_{C_2} f(x,y,z)ds = \frac{5}{6}\sqrt{5} + \frac{3}{2}.$

Exercise 9.

1. *Evaluate* $\int_C \sqrt{x+2y}\, ds$, *where* C *is*
   (a) *the straight-line segment* $x = t, y = 4t$, *from* $(0,0)$ *to* $(1,4)$.
   (b) $C_1 \cup C_2$; C_1 *is the line segment from* $(0,0)$ *to* $(1,0)$ *and* C_2 *is the line segment from* $(1,0)$ *to* $(1,2)$.
2. *Evaluate* $\displaystyle\int_C \frac{x^2}{y^{4/3}} ds$, *where* C *is the curve* $x = t^2, y = t^3$, *for* $1 \leq t \leq 2$.

Solution for Exercise 9

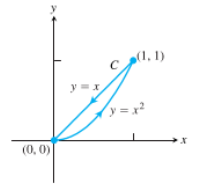
1. (a) $\mathbf{r}(t) = t\mathbf{i} + 4t\mathbf{j}, 0 \le t \le 1 \Rightarrow \frac{dr}{dt} = \mathbf{i} + 4\mathbf{j} \Rightarrow |\frac{dr}{dt}| = \sqrt{17} \Rightarrow$ $\int_c \sqrt{x+2y}\ ds = \int_0^1 \sqrt{t+2(4t)}\sqrt{17}\ dt = \sqrt{17}\int_0^1 \sqrt{9t}\ dt =$ $3\sqrt{17}\int_0^1 \sqrt{t}\ dt = \left[2\sqrt{17}t^{2/3}\right]_0^1 = 2\sqrt{17}$

   (b) $C_1 : \mathbf{r}(t) = t\mathbf{i}, 0 \le t \le 1 \Rightarrow \frac{dr}{dt} = \mathbf{i} \Rightarrow |\frac{dr}{dt}| = 1$; $C_2 : \mathbf{r}(t) = \mathbf{i} + t\mathbf{j}, 0 \le$ $t \le 1 \Rightarrow \frac{dr}{dt} = \mathbf{j} \Rightarrow |\frac{dr}{dt}| = 1$ $\int_c \sqrt{x+2y}\ ds =$ $\int_{c_1} \sqrt{x+2y}\ ds + \int_{c_2} \sqrt{x+2y}\ ds =$ $\int_0^1 \sqrt{t+2(0)}dt + \int_0^2 \sqrt{1+2(t)}\ dt = \int_0^1 \sqrt{t}\ dt + \int_0^2 \sqrt{1+2t}\ dt =$ $\left[\frac{2}{3}t^{2/3}\right]_0^1 + \left[\frac{1}{3}(1+2t)^{2/3}\right]_0^2 = \frac{2}{3} + \left(\frac{5\sqrt{5}}{3} - \frac{1}{3}\right) = \frac{5\sqrt{5}+1}{3}$

2. $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}, 1 \le t \le 2 \Rightarrow \frac{dr}{dt} = 2t\mathbf{i} + 3t^2\mathbf{j} \Rightarrow |\frac{dr}{dt}| =$ $\sqrt{(2t)^2 + (3t^2)^2} = t\sqrt{4+9t^2} \Rightarrow \int_c \frac{x^2}{y^{4/3}} ds = \int_1^2 \frac{(t^2)^2}{(t^3)^{4/3}} \cdot$ $t\sqrt{4+9t^2}dt = \int_1^2 t\sqrt{4+9t^2}dt = \left[\frac{1}{27}(4+9t^2)^{3/2}\right]_1^2 = \frac{80\sqrt{10}-13\sqrt{13}}{48}$

Exercise 10.

Evaluate $\int_C (x + \sqrt{y})ds$ *where* C *is given in the accompanying figure.*



Solution for Exercise 10

$C_1 : \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, 0 \leq t \leq 1 \Rightarrow \frac{dr}{dt} = \mathrm{i} + 2t\mathbf{j} \Rightarrow |\frac{dr}{dt}| = \sqrt{1+4t^2}$

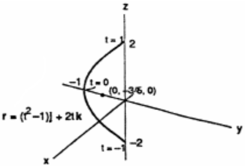
$C_2 : \mathbf{r}(t) = (1-t)\mathbf{i} + (1-t)\mathbf{j}, 0 \leq t \leq 1 \Rightarrow \frac{dr}{dt} = -\mathbf{i} - \mathbf{j} \Rightarrow |\frac{dr}{dt}| = \sqrt{2}$

$\int_c (x+\sqrt{y})ds = \int_{c_1}(x+\sqrt{y})ds + \int_{c_2}(x+\sqrt{y})ds = \int_0^1 (t+\sqrt{t^2})\sqrt{1+4t^2}dt + \int_0^1((1-t)+\sqrt{1-t})\sqrt{2}dt = \int_0^1 2t\sqrt{1+4t^2}dt + \int_0^1(1-t+\sqrt{1-t})\sqrt{2}dt = \left[\frac{1}{6}(1+4t^2)^{3/2}\right]_0^1 + \sqrt{2}\left[t - \frac{1}{2}t^2 - \frac{3}{2}(1-t)^{3/2}\right]_0^1 = \frac{5\sqrt{5}-1}{6} + \frac{7\sqrt{2}}{6} = \frac{5\sqrt{5}+7\sqrt{2}-1}{6}$.

Exercise 11.

1. *Find the area of one side of the "winding wall" standing orthogonally on the curve* $y = x^2, 0 \leq x \leq 2$, *and beneath the curve on the surface* $f(x, y) = x + \sqrt{y}$.
2. *Find the area of one side of the "wall" standing orthogonally on the curve* $2x + 3y = 6, 0 \leq x \leq 6$, *and beneath the curve on the surface* $f(x, y) = 4 + 3x + 2y$.
3. Center of mass of a curved wire : *A wire of density* $\delta(x, y, z) = 15\sqrt{y + 2}$ *lies along the curve* $\mathrm{r}(t) = (t^2 - 1)\mathrm{j} + 2t\mathrm{k}, -1 \leq t \leq 1$. *Find its center of mass. Then sketch the curve and center of mass together.*

1. $y = x^2, 0 \le x \le 2 \Rightarrow \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, 0 \le t \le 2 \Rightarrow \frac{dr}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow |\frac{dr}{dt}| = \sqrt{1+4t^2} \Rightarrow A = \int_c f(x,y)ds = \int_c (x+\sqrt{y})ds = \int_0^2 (t+\sqrt{t^2})\sqrt{1+4t^2}dt = \int_0^2 2t\sqrt{1+4t^2}dt = \left[\frac{1}{6}(1+4t^2)^{3/2}\right]_0^2 = \frac{17\sqrt{17-1}}{6}$
2. $2x + 3y = 6, 0 \le x \le 6 \Rightarrow \mathbf{r}(t) = t\mathbf{i} + (2 - \frac{2}{3}t)\mathbf{j}, 0 \le t \le 6 \Rightarrow \frac{dr}{dt} = \mathbf{i} - \frac{2}{3}\mathbf{j} \Rightarrow |\frac{dr}{dt}| = \frac{\sqrt{13}}{3} \Rightarrow A = \int_c f(x,y)ds = \int_c (4+3x+2y)ds = \int_0^6 (4+3t+2(2-\frac{2}{3}t))\frac{\sqrt{13}}{3}dt = \frac{\sqrt{13}}{3}\int_0^6 (8+\frac{5}{3}t)dt = \frac{\sqrt{13}}{3}[8t+\frac{5}{6}t^2]_0^6 = 26\sqrt{13}$
3. $\mathbf{r}(t) = (t^2-1)\mathbf{j} + 2t\mathbf{k}, -1 \le t \le 1 \Rightarrow \frac{dr}{dt} = 2t\mathbf{j} + 2\mathbf{k} \Rightarrow |\frac{dr}{dt}| = 2\sqrt{t^2+1}; M = \int_c \delta(x,y,z)ds = \int_{-1}^1 (15\sqrt{(t^2-1)+2})(2\sqrt{t^2+1})dt = \int_{-1}^1 30(t^2+1)dt = \left[30\left(\frac{t^3}{3}+t\right)\right]_{-1}^1 = 60\left(\frac{1}{3}+1\right) = 80; M_{xz} = \int_c y\delta(x,y,z)ds = \int_{-1}^1 (t^2-1)[30(t^2+1)]dt = \int_{-1}^1 30(t^4-1)dt = \left[30\left(\frac{t^5}{5}-t\right)\right]_{-1}^1 = 60\left(\frac{1}{5}-1\right) = -48 \Rightarrow \overline{y} = \frac{M_{xz}}{M} = -\frac{48}{80} = -\frac{3}{5}; M_{yz} = \int_c x\delta(x,y,z)ds = \int_c 0\ \delta\ ds = 0 \Rightarrow \overline{x} = 0; \overline{z} = 0$ by symmetry (since δ is independent of z) $\Rightarrow (\overline{x},\overline{y},\overline{z}) = (0, -\frac{3}{5}, 0)$



Exercise 12.

1. Mass of wire with variable density : *Find the mass of a thin wire lying along the curve* $\mathrm{r}(t) = \sqrt{2}t\mathrm{i} + \sqrt{2}t\mathrm{j} + (4 - t^2)\mathrm{k}, 0 \leq t \leq 1$, *if the density is*
   (a) $\delta = 3t$;
   (b) $\delta = 1$.
2. Center of mass of wire with variable density : *Find the center of mass of a thin wire lying along the curve* $\mathrm{r}(t) = t\mathrm{i} + 2t\mathrm{j} + (2/3)t^3/\mathrm{k}, 0 \leq t \leq 2$, *if the density is* $\delta = 3\sqrt{5 + t}$.

Solution for Exercise 12

1. $\mathbf{r}(t) = \sqrt{2}t\mathbf{i} + \sqrt{2}t\mathbf{j} + (4 - t^2)\mathbf{k}, 0 \le t \le 1 \Rightarrow \frac{dr}{dt} = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} - 2t\mathbf{k} \Rightarrow |\frac{dr}{dt}| = \sqrt{2+2+4t^2} = 2\sqrt{1+t^2}$;
   (a) $M = \int_c \delta\ ds = \int_0^1 (3t)(\sqrt{1+t^2})dt = \left[2(1+t^2)^{3/2}\right]_0^1 = 2(2^{3/2} - 1) = 4\sqrt{2} - 2$
   (b) $M = \int_c \delta\ ds = \int_0^1 (1)(2\sqrt{1+t^2})dt = \left[t\sqrt{1+t^2} + \ln(t + \sqrt{1+t^2})\right]_0^1 = \left[\sqrt{2} + \ln(1+\sqrt{2})\right] - (0 + \ln 1) = \sqrt{2} + \ln(1+\sqrt{2})$
2. $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + \frac{2}{3}t^{3/2}\mathbf{k}, 0 \le t \le 2 \Rightarrow \frac{dr}{dt} = \mathbf{i} + 2\mathbf{j} + t^{1/2}\mathbf{k} \Rightarrow |\frac{dr}{dt}| = \sqrt{1+4+t} = \sqrt{5+t}$; $M = \int_c \delta\ ds = \int_0^2 (3\sqrt{5+t})(\sqrt{5+t})dt = \int_0^2 3(5+t)dt = \left[\frac{3}{2}(5+t)^2\right]_0^2 = \frac{3}{2}(7^2 - 5^2) = \frac{3}{2}(24) = 36$; $M_{yz} = \int_c\ x\delta\ ds = \int_0^2 t[3(5+t)]dt = \int_0^2 (15t + 3t^2)dt = \left[\frac{15}{2}t^2 + t^3\right]_0^2 = 30 + 8 = 38$; $M_{xz} = \int_c\ y\delta\ ds = \int_0^2 2t[3(5+t)]dt = 2\int_0^2 (15t + 3t^2)dt = 76$; $M_{xy} = \int_c\ z\delta\ ds = \int_0^2 \frac{2}{3}t^{3/2}[3(5+t)]dt = \int_0^2 (10t^{3/2} + 2t^{5/2})dt = \left[4t^{5/2} + \frac{4}{7}t^{7/2}\right]_0^2 = 4(2)^{5/2} + \frac{4}{7}(2)^{7/2} = 16\sqrt{2} + \frac{32}{7}\sqrt{2} = \frac{144}{7}\sqrt{2} \Rightarrow \overline{x} = \frac{M_{yz}}{M} = \frac{38}{36} = \frac{19}{18}, \overline{y} = \frac{M_{xz}}{M} = \frac{76}{36} = \frac{19}{9}$, and $\overline{z} = \frac{M_{xy}}{M} = \frac{144\sqrt{2}}{7.36} = \frac{4}{7}\sqrt{2}$

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