Recurrence Relations and Their Solutions (Problem : Tower of Hanoi)

P. Sam Johnson

National Institute of Technology Karnataka (NITK)
Surathkal, Mangalore, India

The Tower of Hanoi

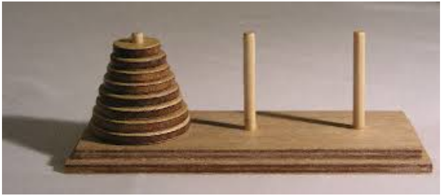
In the lecture, we discuss the problem called "The Tower of Hanoi". Its solution uses the idea of **recurrence**, in which the solution to the problem depends on the solutions to smaller instances of the same problem.

The puzzle was invented by the French mathematician **Edouard Lucas** in 1883. Lucas is known for his study of the Fibonacci sequence.

Edouard Lucas (1842-1891)

Problem

There are three pegs (stands) and eight disks. Initially all eight disks are stacked in decreasing size on one of three pegs.



The **objective** is to transfer the entire disks to one of the other pegs with the following conditions (**Lucas's rules**):

1. moving only one disk at a time, and
2. never moving a larger one onto a smaller.

Now the question is : How many moves are necessary and sufficient to perform the task?

Formation of Recurrence Relation

Let us consider the case if there are n disks.

It is advantageous **to look at small cases first**. It is easy to see how to transfer a tower that contains only one or two disks. And a small amount of experimentation shows how to transfer a tower of three.

We denote T_n the minimum number of moves that will transfer n disks from one peg to another under Lucas's rules.

- Clearly $T_0 = 0$, because **no moves at all needed** to transfer a tower of "0" disks.
- $T_1 = 1$ (the middle peg is not at all used to transfer "1" disk)
- $T_2 = 3$ (transfer the top disk to the middle peg, then move the second, then bring the smaller disk onto it).

How can we transfer a tower of n disks ?

We discuss the case of transfering a tower of 3 disks :

1. transfer the top 2 disks to the middle peg (requiring 3 moves)
2. move the largest disk to the third peg (requiring 1 move)
3. transfer the other two onto the largest disk (requiring 3 moves).

This gives us a clue of transfering n disks in general :

1. transfer the top $n-1$ smaller disks to the middle peg (requiring T_{n-1} moves)
2. move the largest disk to the third peg (requiring 1 move)
3. transfer the $n-1$ smallest onto the largest disk (requiring T_{n-1} moves).

How many moves are necessary and sufficient to perform the task?

Thus we can transfer n disks ($n > 0$) in **at most** $2T_{n-1} + 1$ moves :

$$T_n \leq 2T_{n-1} + 1, \qquad \text{for} \quad n > 0. \tag{1}$$

That is, $2T_{n-1} + 1$ moves suffice. **Are $2T_{n-1} + 1$ moves necessary?**

- At some point we must move the largest disk to the third peg.
- When we do, the $n - 1$ smallest must be on the middle peg, and it has taken **at least** T_{n-1} moves.
- At least 1 move is required to move the largest disk.
- After moving the largest disk, we again require at least T_{n-1} moves.

Hence

$$T_n \geq 2T_{n-1} + 1, \qquad \text{for} \quad n > 0. \tag{2}$$

Recurrence Relation

From (1) and (2), together with the trivial solution for $n = 0$, yield

$$\begin{aligned} T_0 &= 0 \qquad \text{(boundary value)} \\ T_n &= 2T_{n-1} + 1 \quad \text{for} \quad n > 0. \end{aligned}$$

The above set of equalities is called a **recurrence (recursion) relation.** Note that the general value T_n is in terms of earlier ones T_{n-1}.

How do we solve a recurrence relation?

One way is **to guess** the correct solution, then to prove that our guess is correct.

Principle of Mathematical Induction

And our best hope for guessing the solution is **to look at small cases:**

n	0	1	2	3	4	5	6
T_n	0	1	$(2x1)+1$ $=3$	$(2x3)+1$ $=7$	$(2x7)+1$ $=15$	$(2x15)+1$ $=31$	$(2x31)+1$ $=63$

It looks as if

$$T_n = 2^n - 1, \qquad \text{for} \quad n \geq 0. \tag{3}$$

From the table, the solution (called, **"closed form"** for T_n) is correct for $n \leq 6$.

When n is large, is the closed form given in (3) correct?

We shall use a method, called **principle of mathematical induction**, to prove "some statement about the integer n is true for all $n \geq n_0$.

Principle of Mathematical Induction

Mathematical induction is a general way to prove that some statement about the integer n is true for all $n \geq n_0$.

1. First we prove the statement when n has its smallest value, n_0 (called the **basis**).
2. Assuming that it has already been proved for all values between n_0 and $n - 1$, including both n_0 and $n - 1$.
3. We prove the statement for $n > n_0$.

Such a proof gives infinitely many results with only a finite amount of work.

Exercise

1. *By the principle of mathematical induction, prove that $T_n = 2^n - 1$ for $n \geq 0$. Here T_n is the recurrence solution of the problem of "Tower of Hanoi".*

Simple solution for T_n : Adding 1 to both sides of the equations $T_0 = 0$ and $T_n = 2T_{n-1} + 1$ for $n > 0$ and letting $u_n = T_n + 1$, we get $u_0 = 1$ and $u_n = 2u_{n-1}$ for $n > 0$. Hence $u_n = 2^n$. Thus $T_n = 2^n - 1$, for $n \geq 0$.

In finding a closed-form expression (recurrence solution) for some quantity of interest like T_n we go through three stages :

- Look at small cases.
- Find and prove a mathematical expression (recurrence relation) for the quantity of interest.
- Find and prove a closed form (recurrence solution) for our mathematical expression.

Exercises

2. *Find the shortest sequence of moves that transfers a tower of n disks from the left peg A to the right peg B, if direct moves between A and B are disallowed. (Each move must be to or from the middle peg. As usual, a larger disk must never appear above a smaller one.)*
3. *Show that, in the process of transferring a tower under the restrictions of the preceding exercise, we will actually encounter every properly stacked arrangement of n disks on three pegs.*
4. *Are there any starting and ending configurations of n disks on three pegs that are more than $2^n - 1$ moves apart, under Lucas's original rules?*

Exercise

5. *Let Q_n be the minimum number of moves needed to transfer a tower of n disks from A to B if all moves must be clockwise – that is, from A to B, or from B to the other peg, or from the other peg to A. Also let R_n be the minimum number of moves needed to go from B back to A under this restriction. Prove that*

$$Q_n = \begin{cases} 0 & n = 0 \\ 2R_{n-1} + 1 & n > 0 \end{cases}$$

and

$$R_n = \begin{cases} 0 & n = 0 \\ Q_n + Q_{n-1} + 1 & n > 0. \end{cases}$$

Exercise

6. *A* Double Tower of Hanoi *contains* $2n$ *disks of* n *different sizes, two of each size. As usual, we're required to move only one disk at a time, without putting a larger one over a smaller one.*

   (a) *How many moves does it take to transfer a double tower from one peg to another, if disks of equal size are indistinguishable from each other?*

   (b) *What if we are required to reproduce the original top-to-bottom order of all the equal-size disks in the final arrangement?*

Tower of Brahma

Legend (a traditional story sometimes popularly regarded as historical but not authenticated) has it that there is a temple near Hanoi, in Vietnam. In this temple there are monks who are in the process of solving a giant puzzle consisting of 64 golden rings of different sizes on three diamond needles. Gold placed these golden rings on the first needle and ordained that the monks should transfer them to the third, according to Lucas's rule. The monks reportedly work day and night at their task. When they finish, the world will end!

If the legend were true, and if the monks were able to move rings at a rate of 1 per second, it would take them $2^{64} - 1$ seconds or roughly 585 billion years. The universe is currently about 13.7 billion years old.

Lucas's original puzzle is a bit more practical. It requires $2^8 - 1 = 255$ moves, which takes about four minutes for the quick of hand.

References

1. **Graham, Knuth and Patashnik**, *"Concrete Mathematics – A Foundation for Computer Science"*, Pearson Education.
2. **Marko Petkovsek, Herbert S. Wilf and Doron Zeilberger**, "$A = B$", AK Peters Ltd., Wellesley, Massachusetts.
3. **Herbert S. Wilf**, *"Generatingfunctionology"*, Third Edition, AK Peters Ltd., Wellesley, Massachusetts.