Rank-Factorization of a Matrix

P. Sam Johnson

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Aim of the lecture

The aim of the lecture is to discuss full rank matrices and factorization of every non-null matrix as a product of two full rank matrices.

Several nice properties of matrices which are of full rank (either full row rank or full column rank) are discussed.

Definition

Let A be a $m \times n$ matrix. Then the **column space** *of A is $\mathcal{C}(A)$ is*

$$\mathcal{C}(A) := \{Ax : x \in F^n\}$$

and the **row space** *of A is*

$$\mathcal{R}(A) := \{y^T A : y \in F^m\}.$$

- We call $dim(\mathcal{R}(A))$ the **row rank** of A and $dim(\mathcal{C}(A))$ the **column rank** of A.
- We refer to a basis of $\mathcal{C}(A)$ consisting of columns of A as a **column basis**. A **row basis** is defined similarly.

If A has column rank r, then

- any r linearly independent columns of A form a basis for $\mathcal{C}(A)$,
- every maximal linearly independent set of columns of A contains exactly r vectors,
- any r columns of A which generate $\mathcal{C}(A)$ form a basis of $\mathcal{C}(A)$.

Theorem

For any matrix A, the row rank of A equals the column rank of A.

Definition

The **rank** *of a matrix A is the common value of the row rank of A and the column rank of A and is denoted by $\rho(A)$.*

Definition

An $m \times n$ matrix A is said to be of **full row rank** *if its rows are linearly independent, that is, it its rank is m. Similarly A is said to be of* **full column rank** *if its columns are linearly independent.*

A **left inverse** *of a matrix A is any matrix B such that $BA = I$. A* **right inverse** *of A is any matrix C such that $AC = I$.*

A matrix B is said to be an **inverse** *of A if it is both a left inverse and a right inverse of A.*

Theorem

Let A be a $m \times n$ matrix over F. Then the following statements are equivalent.

1. *A has a right inverse.*
2. $XA = 0 \Rightarrow X = 0$.
3. *A is of full row rank.*

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Definition

Let A be a $m \times n$ matrix with rank $r \geq 1$. Then (P, Q) is said to be a **rank-factorization** *of A if P is of order $m \times r$, Q is of order $r \times n$ and $A = PQ$.*

Theorem

Every non-null matrix has a rank-factorization.

Proof. Let A be a $m \times n$ matrix with rank r.

Let $B = [x_1 : x_2 : \cdots : x_r]$ be an $m \times r$ matrix whose columns form a basis of $\mathcal{C}(A)$. Then for each $j = 1, 2, \ldots, n$, each column of A, A_{*j} is a linear combination of the columns of B, so there exists an $r \times 1$ vector y_j such that $A_{*j} = By_j$.

Now

$$\begin{aligned} A &= [A_{*1} : \cdots : A_{*n}] \\ &= [By_1 : \cdots : By_n] \\ &= B[y_1 : \cdots : y_n] \\ &= BC \end{aligned}$$

where $C = [y_1 : \cdots : y_n]$.

- **A null matrix cannot have a rank-factorization** since there cannot be a matrix with 0 rows.
- **Rank-factorization of a matrix is not unique.** The choice of the matrix B is not unique because the columns of B are coming from the column basis of A.
- If (B, C) is a rank-factorization of A, then (C^T, B^T) is a rank-factorization of A^T.

When a factorization is a rank-factorization?

Theorem

Let $A = PQ$ where P is a $m \times k$ matrix and Q a $k \times n$ matrix. Then the rank of A is at most k.

Moreover, the following are equivalent:

- *the rank of A is k,*
- *(P, Q) is a rank-factorization of A,*
- *P is of full column rank and Q is of full row rank,*
- *the columns of P form a basis of $\mathcal{C}(A)$,*
- *the row of Q form a basis of $\mathcal{R}(A)$.*

Corollary

If (P, Q) is a rank-factorization of A then $\mathcal{C}(P) = \mathcal{C}(A)$, $\mathcal{R}(Q) = \mathcal{R}(A)$ and $\mathcal{N}(Q) = \mathcal{N}(B)$.

Theorem

If $A = A^2$, rank of A equals trace of A.

Proof. The result is trivial if the rank r of A is 0, so let $r \geq 1$.

Let (P, Q) be a rank-factorization of A. Then $PQPQ = PQ = PI_rQ$.

Since P is of full column rank and Q is of full row rank, left and cancellation laws are applied, we get $PA = I_r$.

Hence rank of $A = r = tr(I_r) = tr(QP) = tr(PQ) = tr(A)$.

Finding a rank-factorization of a matrix A of rank r is easy when A is represented in the following nice form.

Theorem

Let A be an $m \times n$ matrix of rank $r \geq 1$. Then there exist permutation matrices P and Q such that

$$A = P \begin{pmatrix} B & BC \\ DB & DBC \end{pmatrix} Q$$

where B is non-singular matrix of order r and, C and D are some matrices of orders $r \times (n-r)$ and $(m-r) \times r$ respectively.

When a matrix A in the above form, can be factorized as $A = P_1 Q_1$ where

$$P_1 = P \begin{pmatrix} B \\ DB \end{pmatrix} \text{ and } Q_1 = \begin{pmatrix} I_r & : & C \end{pmatrix} Q.$$

Since P_1 is of order $m \times r$, it follows that (P_1, Q_1) is a rank-factorization of A.

References

- S. Kumaresan, "*Linear Algebra - A Geometric Approach*", PHI Learning Pvt. Ltd., 2011.
- A. Ramachandra Rao and P. Bhimasankaram, "*Linear Algebra*", Hindustan Book Agency, 2000.