Vector Spaces and Subspaces

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Overview

Two operations, addition and scalar multiplication (multiplication by real scalars) are defined on a set. In the set we can add any two vectors, and we can multiply vectors by real scalars. The set becomes a vector space if eight axioms are satisfied.

The following are discussed in the lecture.

1. Formal definition of a **real vector space** is given, with examples.
2. A **subspace** is a subset of a vector space which is "closed" under additon and scalar mutiplication. For a given matrix of order $m \times n$, two interesting subspaces (**column space** and **null space**) are defined in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively.
3. Finally, a result connecting general solutions of homogeneous system ($Ax = b$) and non-homogeneous system ($Ax = 0$), is given. The result is helpful in writing down the **general solution of the non-homogeneous system**.

Introduction

The space $\mathbb{R}^n$ consists of all column vectors with n components. (The components are real numbers.) The space $\mathbb{R}^2$ is represented by the usual xy-plane; the two components of the vector become the x and y coordinates of the corresponding point.

$\mathbb{R}^3$ is equally familiar, with the three components giving a point in three-dimensional space.

The one-dimensional space $\mathbb{R}^1$ is a line. The valuable thing for linear algebra is that the extension to n dimensions is so straightforward; for a vector in seven-dimensional space $\mathbb{R}^7$ we just need to know the seven components, even if the geometry is hard to visualize.

Within these spaces, and within all vector spaces, two operations are possible: **We can add any two vectors, and we can multiply vectors by real scalars**. For the spaces $\mathbb{R}^n$ these operations are done a component at a time.

Definition : Vector Space (also called, Linear Space)

A **real vector space** X is a set of "vectors" together with rules for vector addition and multiplication by real numbers. The addition and multiplication (by real numbers) must produce vectors that are within the space, and they must satisfy the following **eight axioms** for all $\alpha, \beta \in \mathbb{R}$ and $x, y, z \in X$:

1. $(x + y) + z = x + (y + z)$ (Associativity of $+$)
2. there exists an elements 0 of X such that $x + 0 = x$ for all $x \in X$ (Existence of 0)
3. for each $x \in X$ there exists an element $-x$ in X such that $x + (-x) = 0$ (Existence of negative)
4. $x + y = y + x$ (Commutativity of $+$)
5. $(\alpha + \beta)x = \alpha x + \beta x$ (Distributivity)
6. $\alpha(x + y) = \alpha x + \alpha y$ (Distributivity)
7. $\alpha(\beta x) = (\alpha\beta)x$
8. $1.x = x$.

Examples of Vector Spaces

1. The space $\mathbb{R}^n$ consists of all column vectors with n components.
2. The infinite-dimensional space $\mathbb{R}^\infty$.
3. The space of 3 by 2 matrices. In this case the "vectors" are matrices! We can add two matrices, and $A + B = B + A$, and there is a zero matrix, and so on. This space is almost the same as $\mathbb{R}^6$. (The six components are arranged in a rectangle instead of a column.)
4. The space of m by n matrices.
5. The space of functions f defined on a fixed interval, say $0 \leq x \leq 1$. The vectors are functions, and again the dimension is infinite - in fact, it is a larger infinity than for $\mathbb{R}^\infty$.

Exercise

Construct a subset of the xy-plane $\mathbb{R}^2$ *that is*

1. *closed under vector addition and subtraction, but not scalar multiplication.*
2. *closed under scalar multiplication but not under vector addition.*

Subspaces

Geometrically, think of the usual three-dimensional $\mathbb{R}^3$ and choose any plane through the origin. That plane is a vector space in its own right.

If we multiply a vector in the plane by 3, or -3, or any other scalar, we get a vector which lies in the same plane. If we add two vectors in the plane, their sum stays in the plane.

This plane illustrates one of the most fundamental ideas in the theory of linear algebra; it is a subspace of the original space $\mathbb{R}^3$.

Subspaces

A **subspace** of a vector space is a nonempty subset that satisfies two requirements:

1. If we **add** any vectors x and y in the subspace, their sum $x + y$ is in the subspace.
2. If we **multiply any vector** x in the subspace **by any scalar** c, the multiple cx is still in the subspace.

In other words, a subspace is a subset which is "**closed**" under additon and scalar mutiplication. Those operations follow the rules of the host space, without taking us outside the subspace.

There is **no need** to verify the eight required properties, because they are satisfied in the larger space and will automatically be satisfied in every subspace.

Notice in particular that the **zero vector** will belong to every subspace.

Subspaces

The most extreme posibility for a subspace is to contain only one vector, the zero vector. It is a "zero-dimensional space," **containing only the zero vector**. This is **the smallest possible vector space**. Note that the empty set is not allowed.

At the other extreme, **the largest subspace is the whole of the original space** - we can allow every vector into the subspace.

If the original space is $\mathbb{R}^3$, then the possible subspaces are easy to describe: $\mathbb{R}^3$ itself, any plane through the origin, any line through the origin, or the origin (the zero vector) alone.

Exercise

Which of the following are subspaces of $\mathbb{R}^\infty$ *?*

1. *All sequences like* $(1, 0, 1, 0, \ldots)$ *that include infinitely many rows.*
2. *All sequences* $(x_1, x_2, x_3, \ldots)$ *with* $x_j = 0$ *for some point onward.*
3. *All convergent sequences.*
4. *All geometric progression* $(x_1, kx_1, k^2x_1, \ldots)$ *allowing all* k *and* x_1*.*

Smallest Subspace Containing a Set

The distinction between a subset and a subspace is made clear by examples: Consider all vectors whose components are positive or zero. If the original space is the xy-plane $\mathbb{R}^2$, then this subset is the first quadrant; the coordinates satisfy $x \geq 0$ and $y \geq 0$. It is not a subspace, even though it contains zero and addition does leave us within the subset.

If $c = -1$ and $x = (1, 1)$, the multiple $cx = (-1, -1)$ is in the third quadrant instead of the first. If we include the third quadrant along with the first, then scalar multiplication is all right; every mutiple cx will staty in this subset, however the addition of $(1, 2)$ and $(-2, -1)$ gives a vector $(-1, 1)$ which is not in either quadrant.

The **smallest subspace** containing the first quadrant is the whole space $\mathbb{R}^2$.

If we start from the vector space of 3 by 3 matrices, then one possible subspace is **the set of lower triangular matrices**.

Another is **the set of symmetric matrices**. In both cases, the sums $A + B$ and the multiples cA inherit the properties of A and B. They are lower triangular if A and B are lower triangular, and they are symmetric if A and B are symmetric.

Of course, the zero matrix is in both **subspaces**.

Exercise

What is the smallest subspace of 3×3 *matrices that contains all symmetric matrices* **and** *all lower triangular matrices? What is the largest subspace that is contained in* **both** *of those subspaces?*

Column space - An Example of a Subspace

We now come to the key examples of subspaces. They are tied directly to a $m \times n$ matrix A, and they give information about the system $Ax = b$.

The **column space** contains all linear combinations of the columns of A and it is denoted by $C(A)$. The system $Ax = b$ is solvable iff the vector b can be expressed as a combination of the columns of A. Then b is in the column space.

Example

The matrices $A = \begin{pmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 2 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 2 & 4 \end{pmatrix}$ *have the same column spaces.*

Note that the third column of B is the sum of first and second columns of B.

Column space is a subspace of $\mathbb{R}^m$.

Suppose b and b' lie in the column space, so that $Ax = b$ for some x and $A' = b'$ for some x'; x and x' just give the combinations which produce b and b'.

Then $A(x + x') = b + b'$, so that $b + b'$ is also a combination of the columns. The attainable vectors are closed under addition, and the first requirement for a subspace is met.

If b is in the column space, so is any multiple cb. If some combination of columns produces b (say $Ax = b$), then multiplying every coefficient in the combination by c will produce cb. In other words, $A(cx) = cb$.

The smallest possible column space comes from the zero matrix $A = 0$. The only vector in its column space (the only combination of the columns) is $b = 0$, and no other choice of b allows us to solve $0x = b$.

Example

Let $A = \begin{pmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{pmatrix}$. A restatement of the system $Ax = b$ is written as follows : $u \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} + v \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

The subset of attainable right-hand sides b is the set of all combinations of the columns of A.

One possible right side is the first column itself; the weights are $u = 1$ and $v = 0$.

Another possiblity is the second column: $u = 0$ and $v = 1$. A third is the right side $b = 0$; the weights are $u = 0, v = 0$ (and with that trivial choice, the vector $b = 0$ will be attainable no matter what the matrix is).

Column Space is Full.

At the other extreme, suppose A is the 5 by 5 **identity matrix**. Then the column space is the whole of $\mathbb{R}^5$; the five columns of the identity matrix can combine to produce any five-dimensional vector b.

This is not at all special to the identity matrix.

Any 5 by 5 matrix which is nonsingular will have the whole of $\mathbb{R}^5$ as its column space. For such a matrix we can solve $Ax = b$ by Gaussian elimination; there are five pivots.

Therefore every b is in the column space of a **nonsingular matrix**.

When is $Ax = b$ solvable?

Now we have to consider all combinations of the two columns, and we describe the result geometrically: $Ax = b$ can be solved iff b lies in the plane that is spanned by the two column vectors. This is the thin set of attainable b.

If b lies off the plane, then it is not a combination of the two columns. In this case $Ax = b$ has no solution.

What is important is that this plane is not just a subset of $\mathbb{R}^3$; it is a subspace.

For what value of b, is the system $Ax = b$ solvable?

The equation $Ax = b$ can be solved iff b lies in the column space of A.

Nullspace : Another Example of a Subspace

The **nullspace of a matrix** consists of all vectors x such that $Ax = 0$ (i.e., the set of solutions to $Ax = 0$). It is denoted by $N(A)$.

- If $Ax = 0$ and $Ay = 0$, then $A(x + y) = 0$.
- If $Ax = 0$, then $A(cx) = 0$.

As both requirement are satisfied, $N(A)$ **is a subspace of** $\mathbb{R}^n$**.**

Note that both requirements fail if the right-hand side is not zero!

Solving $Ax = 0$ and $Ax = b$

Consider a system of m linear equations with n unknowns

$$Ax = b. \tag{1}$$

When $b = 0$, it is called **homogeneous system**; otherwise **nonhomogeneous**.

The system

$$Ax = 0 \tag{2}$$

is called the homogeneous system associated with 1. The above system always has a solution 0 (the **zero column vector**), called **zero or trivial solutions**.

General Solution of Homogeneous System

The fundamental relationship between the systems 1 and 2 follows :

Theorem

Suppse u is a particular solution of the nonhomogeneous system 1 and supposed W is the general solution of the associated homogeneous system 2. Then

$$u + W = \{u + w : w \in W\}$$

is the general solution of the non-homogeneous system 1.

We emphasize that the above theorem is of theoretical interest and does not help us to obtain explicit solutions of the system 1. But by the method of (Gaussian) elimination, the general solution of the non-homogeneous system can be found.

Example : Simple System – One Equation and One Unknown

Consider the simple 1×1 system $ax = b$, one equation and one unknown. There are three possibilities.

1. Suppose $a \neq 0$. The system has **unique solution** b/a.
2. Suppse $a = 0$ but $b \neq 0$. Then $0x = b$ has **no solution**. The **column space** of 1×1 zero matrix contains only $b = 0$.
3. Suppse both a and b are zero. Then the system $0x = 0$ has **infinitely many solutions**. The **nullspace** contains all x. A **particular solution** is $x_p = 0$, and the **complete solution** is $x_p + (\texttt{any}\ \ x) = 0 + (\texttt{any}\ \ x)$.

I hope you can solve (**happily, easily**) all problems in problem sheet-1.