Hermite Interpolation

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Overview

Osculating polynomials generalize both the Taylor polynomials and the Lagrange polynomials.

Suppose that we are given $n+1$ distinct numbers $x_0, x_1, \ldots, x_n$ in $[a, b]$ and nonnegative integers $m_0, m_1, \ldots, m_n$, and

$$m = \max\{m_0, m_1, \ldots, m_n\}.$$

Note that the (unknown) function f is m_i-times differentiable at x_i.

The osculating polynomial approximating a function $f \in C^m[a, b]$ at x_i, for each $i = 0, 1, \ldots, n$, is the polynomial of least degree with the property that it agrees with the function f and all its derivatives of order less than or equal to m_i at x_i.

Osculating Polynomials

That is, the osculating polynomial $P(x)$ approximating a function $f \in C^m[a, b]$ satisfies the following: For each $i = 0, 1, 2, \ldots, n$

1. $P(x_i) = f(x_i)$
2. $P^k(x_i) = f^k(x_i)$, for all $1 \leq k \leq m_i$.

$P(x)$ is **the unique polynomial of least degree with the above properties.**

Special Cases:

1. When $n = 0$, the osculating polynomial P approximating f is the m_0th Taylor polynomial for f at x_0.
2. When $m_i = 0$ for each i, the osculating polynomial P approximating f is the nth Lagrange interpolating polynomial for f at $x_0, x_1, \ldots, x_n$.

Hermite Polynomials

The case when $m_i = 1$, for each $i = 0, 1, \ldots, n$, gives the **Hermite polynomials.**

For a given function f, these polynomials agree with f at $x_0, x_1, \ldots, x_n$.

In addition, since their first derivatives agree with those of f, they have the same *shape* as the function at $(x_i, f(x_i))$ in the sense that the **tangent lines** to the polynomial and to the function agree.

We restrict our attention to Hermite polynomials.

Hermite Polynomials

The interpolating polynomials that we have considered so far make use of a certain number of function values. We now derive an interpolation polynomial in which both the function values and its first derivative values are to be assigned at each point of interpolation.

The interpolation problem can be stated as follows.

Given a set of data points (x_i, y_i, y_i'), $i = 0, 1, \ldots, n$, determine a polynomial of least degree, which is denoted by $H_{2n+1}(x)$ such that for all $i = 0, 1, \ldots, n$, we have

$$H_{2n+1}(x_i) = y_i \quad \text{and} \tag{1}$$

$$H'_{2n+1}(x_i) = y'. \tag{2}$$

The polynomial $H_{2n+1}(x)$ is called **Hermite's interpolation polynomial**.

Since we have $2n+2$ conditions the number of coefficients to be determined is $2n+1$ and hence the degree of $H_{2n+1}(x)$ is $2n+1$. The required polynomial $H_{2n+1}(x)$ can be written as

$$H_{2n+1}(x) = \sum_{i=0}^{n} A_i(x) y_i + \sum_{i=0}^{n} B_i(x) y_i'$$

where $A_i(x)$ and $B_i(x)$ are polynomials of degree $\leq 2n+1$.
Using (1) in (2) we obtain the following conditions.

$$(i)\ A_i(x_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$$(ii)\ B_i(x_j) = 0 \quad \text{for all } i \text{ and } j$$

$$(iii)\ A_i'(x_j) = 0 \quad \text{for all } i \text{ and } j$$

$$(iv)\ B_i'(x_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Hermite Polynomials

Since $A_i(x)$ and $B_i(x)$ are polynomials of degree $\leq 2n+1$ we write

$$A_i(x) = u_i(x)\ell_i^2(x) \quad \text{and} \tag{3}$$

$$B_i(x) = v_i(x)\ell_i^2(x), \tag{4}$$

where

$$\ell_i(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)}.$$

Note that $\ell_i(x)$ are Lagrange's interpolation polynomials, and

$$\ell_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Hermite Polynomials

$\ell_i^2(x)$ is a polynomial of degree $2n$ and $A_i(x)$ and $B_i(x)$ are polynomials of degree 1 we see that $u_i(x)$ and $v_i(x)$ are polynomials of degree 1.

$$\text{Let} \quad u_i(x) = a_i x + b_i$$
$$v_i(x) = c_i x + d_i.$$

Thus

$$A_i(x) = (a_i x + b_i)\ell_i^2(x) \tag{5}$$
$$B_i(x) = (c_i x + d_i)\ell_i^2(x). \tag{6}$$

Using conditions (3) and (4) in (5) we obtain

$$
\begin{aligned}
a_i x_i + b_i &= 1 \\
c_i x_i + d_i &= 0 \\
a_i + 2\ell_i'(x_i) &= 0 \\
c_i &= 1 \\
\text{Hence we obtain} \quad a_i &= -2\ell_i'(x_i) \\
b_i &= 1 + 2x_i\ell_i'(x_i) \\
c_i &= 1 \\
\text{and} \qquad d_i &= -x_i. \\
\text{Hence (5) becomes, } A_i(x) &= [1 - 2(x - x_i)\ell_i'(x_i)]\ell_i^2(x) \quad \text{and} \\
B_i(x) &= (x - x_i)\ell_i^2(x).
\end{aligned}
$$

Hermite Polynomials

The required Hermite's interpolation polynomial is

$$H_{2n+1}(x) = \sum_{i=0}^{n} A_i(x) y_i + \sum_{i=0}^{n} B_i(x) y_i'$$

$$\text{where} \quad A_i(x) = [1 - 2(x - x_i)\ell_i'(x_i)]\ell_i^2(x)$$

$$B_i(x) = (x - x_i)\ell_i^2(x).$$

Theorem

If $f \in C^1[a,b]$ and $x_0, x_1, \ldots, x_n \in [a,b]$ are distinct, the unique polynomial of least degree agreeing with f and f' at $x_0, x_1, \ldots, x_n$ is the Hermite polynomial of degree at most $2n+1$ given by

$$H_{2n+1}(x) = \sum_{i=0}^{n} A_i(x) y_i + \sum_{i=0}^{n} B_i(x) y_i'$$

where

$$A_i(x) = [1 - 2(x - x_i)\ell_i'(x_i)]\ell_i^2(x) \quad \textit{and} \quad B_i(x) = (x - x_i)\ell_i^2(x).$$

Note that here $\ell_i(x)$ denotes the i^{th} Lagrange's interpolating polynomial of degree n,

$$\ell_i(x) = \frac{(x - x_0)(x - x_1)\ldots(x - x_{i-1})(x - x_{i+1})\ldots(x - x_n)}{(x_i - x_0)(x_i - x_1)\ldots(x_i - x_{i-1})(x_i - x_{i+1})\ldots(x_i - x_n)}.$$

Error Term

$H_{2n+1}(x)$ is the Hermite polynomial of degree at most $2n+1$

1. agreeing with f at $x_0, x_1, \ldots, x_n$, and
2. their first derivatives (of $H_{2n+1}(x)$) agreeing with those of f.

Moreover, if $f \in C^{2n+2}[a, b]$, then

$$f(x) = H_{2n+1}(x) + \frac{(x - x_0)^2 \ldots (x - x_n)^2}{(2n+2)!} f^{(2n+2)}(\xi)$$

for some (generally unknow) ξ in the interval (a, b).

How to find Hermite polynomial?

For large value of n, the Hermite interpolation method is tedious to apply. An explanation is given for three nodes.

Suppose we are given a table containing values of the triplets

$$\left[x_k, f(x_k), f'(x_k)\right], \text{ for } k = 0, 1, 2.$$

Calculate the three Lagrange polynomials (each of degree 2) about

$$\{x_1, x_2\}, \{x_2, x_0\} \text{ and } \{x_0, x_1\},$$

denoted the polynomials by $\ell_0(x), \ell_1(x), \ell_2(x)$.
Calculate their derivates $\ell_0'(x), \ell_1'(x), \ell_2'(x)$.

How to find Hermite polynomial?

The polynomials

$$A_0(x), A_1(x), A_2(x)$$

and

$$B_0(x), B_1(x), B_2(x).$$

are calculated.
Hence the Hermite polynomial of degree 5

$$\begin{aligned} H_5(x) = & A_0(x)y_0 + A_1(x)y_1 + A_2(x)y_2 + \\ & B_0(x)y_0' + B_1(x)y_1' + B_2(x)y_2'. \end{aligned}$$

Finally, we can evaluate an *approximate value of f* at the specified point. Note that the Hermite polynomial H_5 agrees with f and its derivative, at the given nodes x_0, x_1, x_2.

References

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