Numerical Solution of Ordinary Differential Equations (Part - 1)

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Overview

We discuss the following important methods of solving ordinary differential equations of first / second order.

- Picard's method of successive approximations
  (Method of successive integration)
- Taylor's series method
- Euler's method
- Modified Euler's method

Introduction

A number of problems in science and technology can be formulated into differential equations. The analytical methods of solving differential equations are applicable only to a limited class of equations.

Quite often differential equations appearing in physical problems do not belong to any of these familiar types and one is obliged to resort to numerical methods. These methods are of even greater importance when we realize that computing machines are now readily available which reduce numerical work considerably.

Solution of a Differential Equation

The solution of an ordinary differential equation means finding an explicit expression for y in terms of a finite number of elementary functions of x.

Such a solution of a differential equation is known as the **closed** or **finite form of solution**.

In the absence of such a solution, we have numerical methods to calculate approximate solution.

Numerical Solution of Ordinary Differential Equations of First Order

Let us consider the first order differential equation

$$\frac{dy}{dx} = f(x, y) \qquad \text{given} \quad y(x_0) = y_0 \tag{1}$$

to study the various numerical methods of solving such equations.

In most of these methods, we replace the differential equation by a difference equation and then solve it.

These methods yield **solutions** either as a power series in x from which the values of y can be found by direct substitution or a set of values of x and y.

Single-step Methods

The methods of Picard and Taylor series belong to the former class of solutions. In these methods, y in (1) is approximated by a truncated series, each term of which is a function of x.

The information about the curve at one point is utilized and the solution is not iterated.

As such, these are referred to as **single-step methods**.

Step-by-step Methods

The methods of Euler, Runge-Kutta, Milne, Adams-Bashforth etc. belong to the latter class of solutions.

In these methods, the next point on the curve is evaluated in short steps ahead, by performing iterations till sufficient accuracy is achieved. As such, these methods are called **step-by-step methods**.

Euler and Runge-Kutta methods are used for computing y over a limited range of x-values whereas Milne and Adams methods may be applied for finding y over a wider range of x-values.

Therefore Milne and Adams methods require starting values which are found by Picard, Taylor series or Runge-Kutta method.

Initial and Boundary Conditions

An ordinary differential equation of the *n*th order is of the form

$$F\left(x,\ y,\ \frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \ldots,\ \frac{d^ny}{dx^n}\right) = 0.$$

Its general solution contains *n* arbitrary constants and is of the form

$$\phi(x,\ y,\ c_1,\ c_2,\ \ldots,\ c_n) = 0.$$

To obtain its particular solution, n conditions must be given so that the constants $c_1, c_2, \ldots, c_n$ can be determined.

If these conditions are prescribed at one point only (say, x_0), then the differential equation together with the conditions constitute an **initial value problem** of the nth order.

If the conditions are prescribed at two or more points, then the problem is termed as **boundary value problem**.

Picard's Method

Condier the first order differential equation

$$\frac{dy}{dx} = f(x, y) \qquad \text{given} \quad y(x_0) = y_0. \tag{2}$$

It is required to find that particular solution of (2) which assumes the values y_0 when $x = x_0$.

Integrating (2) between limits, we get

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\ dx \quad \Rightarrow \quad y = y_0 + \int_{x_0}^{x} f(x, y)\ dx.$$

This is an integral equation equivalent to (2).

Picard's Method

As a first approximation y_1 to the solution, we put $y = y_0$ in $f(x, y)$ and integrate (2), giving

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx.$$

For a second approximation y_2, we put $y = y_1$ in $f(x, y)$ and integrate (2), giving $y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx$.

Continuing this process, we obtain $y_3, y_4, \ldots, y_n$, where

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx.$$

Hence this method gives a sequence of approximations $y_1, y_2, \ldots$, each giving a better result than the preceding one.

Picard's Method

- Picard's method is of considerable theoretical value, but can be applied only to a limited class of equations in which the successive integrations can be performed easily.
- The method can be extended to simultaneous equations and equations of higher order.

Exercises

1. Using Picard's process of successive approximations, obtain a solution upto the fifty approximation of the equation

$$\frac{dy}{dx} = y + x$$

   such that $y = 1$ when $x = 0$. Check your answer by finding the exact particular solution.

2. Find the value of y for $x = 0.1$ by Picard's method, given that

$$\frac{dy}{dx} = \frac{y - x}{y + x}$$

   such that $y = 1$ when $x = 0$.

Exercises

3. Solve for $y(0.2)$ by Picard's method,

$$\frac{dy}{dx} = y^2 + x$$

   where $y(0) = 0$.

4. Use Picard's method to approximate the value of y when $x = 0.1$ given that $y(0) = 1$ and $\frac{dy}{dx} = 3x + y^2$.

5. Find the solution of

$$\frac{dy}{dx} = 1 + xy$$

   which passes through $(0, 1)$ in the interval $(0, 0.5)$ such that the value of y is correct to 3 decimal places. Use the whole interval as one interval only and take $h = 0.1$.

Exercises

6. Use Picard's method to approximate y when $x = 0.2$ given that $y = 1$ when $x = 0$ and $\frac{dy}{dx} = x - y$.
7. Find an approximate solution of the initial value problem $y' = 1 + y^2, \quad y(0) = 0$ by Picard's method and compare with the exact solution.
8. Find the successive approximate solution of the differential equation $y' = y$, $y(0) = 1$, by Picard's method and compare it with the exact solution.
9. Given the differential equation

$$\frac{dy}{dx} = \frac{x^2}{y^2 + 1}$$

with the initial condition $y = 0$ when $x = 0$. Use Picard's method to obtain y for $0.25, 0.5$ and 1.0 correct to 3 decimal places.

Solving Simultaneous First Order Differential Equations : Picard's Method

Consider the simultaneous differential equations of the type

$$\frac{dy}{dx} = f(x, y, z) \tag{3}$$

and

$$\frac{dz}{dx} = g(x, y, z) \tag{4}$$

with initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$ can be solved by Picard's method.

The first approximate solution of the equations (3) and (4) are given by

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0, z_0)dx$$

$$z_1 = z_0 + \int_{x_0}^{x} g(x, y_0, z_0)dx.$$

Solving Simultaneous First Order Differential Equations : Picard's Method

Similarly, the higher approximations of y and z are given as follows:

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1, z_1)dx, \qquad z_2 = z_0 + \int_{x_0}^{x} g(x, y_1, z_1)dx$$

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2, z_2)dx, \qquad z_3 = z_0 + \int_{x_0}^{x} g(x, y_2, z_2)dx$$

and so on.

Exercises

10. Apply Picard's method to find the second approximations to the values of y and z by corresponding to $x = 0.1$ given that

$$\frac{dy}{dx} = z, \quad \frac{dz}{dx} = x^3(y+z)$$

given that $y = 1, z = \frac{1}{2}$ when $x = 0$.

11. Approximate y and z by using Picard's method for the particular solution of

$$\frac{dy}{dx} = x + z, \quad \frac{dz}{dx} = x - y^2$$

given that $y = 2, z = 1$ when $x = 0$.

Exercises

12. Approximate y and z by using Picard's method for the particular solution of
$$\frac{dy}{dx} = 1 + xyz, \quad \frac{dz}{dx} = x + y + z$$
given that $y = 0, z = 1$ when $x = 0$.

Solving Second / Higher Order Differential Equations : Picard's Method

Consider a second order differential equation

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right).$$

Let $z = \frac{dy}{dx}$. Hence the given equation can be reduced to 2 first order simultaneous differential equations

$$\frac{dy}{dx} = z$$

and

$$\frac{dz}{dx} = f(x, y, z).$$

Exercises

13. Use Picard's method to approximate y when $x = 0.2$ given that $y = 1,\ y' = 0$ when $x = 0$ and

$$y'' = y + xy'.$$

14. Using Picard's method, obtain the second approximation to the solution at $x = 0.2$ of the differential equation

$$\frac{d^2y}{dx^2} = x^3\frac{dy}{dx} + x^3y$$

with $y(0) = 1,\ y'(0) = 1/2$.

Taylor's Series Method

Consider the first order differential equation

$$\frac{dy}{dx} = f(x, y) \qquad \text{given} \quad y(x_0) = y_0. \tag{5}$$

Differentiating (5), we have

$$\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.\frac{dy}{dx} \qquad \Rightarrow \qquad y'' = f_x + f_y f.$$

Differentiating this successively, we can get y''', y^{iv} etc.

Putting $x = x_0$ and $y = y_0$, the values of $y', y'', y''', \dots,$ can be obtained at $x = x_0$.

Taylor's Series Method

Hence the Taylor's series

$$y = y_0 + (x - x_0)(y')_{x=x_0} + \frac{(x - x_0)^2}{2!}(y'')_{x=x_0} + \cdots \qquad (6)$$

gives the values of y for every value of x for which (6) converges.

On finding the value y_1 for $x = x_1$ from (6), y', y'' etc. can be evaluated at $x = x_1$.

Then y can be expanded about $x = x_1$.

In this way, the solution can be extended beyond the range of convergence of series (6).

Main Drawback of Taylor's Series Method

This is a single step method and works well so long as the successive derivatives can be calculated easily.

If $f(x, y)$ is somewhat complicated and the calculation of higher order derivatives becomes tedious, then Taylor's method cannot be used gainfully. This is the main drawback of this method and therefore, has little application for computer programmes.

However, it is useful for finding starting values for the application of powerful methods like Runge-Kutta, Milne and Adams-Bashforth which will be discussed later.

15. Solve
$$\frac{dy}{dx} = x + y, \quad y(0) = 1$$
by Taylor's series method. Hence find the values of y at $x = 0.1$ and $x = 0.2$.
Compare the final result with the value of the explicit solution.
16. Find by Taylor's series method, the values of y at $x = 0.1$ and $x = 0.2$ to five places of decimals from
$$\frac{dy}{dx} = x^2 y - 1, \quad y(0) = 1.$$
17. Using Taylor's series method, find y at $x = 1.1$ and 0.2 by solving
$$\frac{dy}{dx} = x^2 + y^2, \quad y(1) = 2.3.$$

Exercises

18. Evaluate the solution of the following problem by the first four terms of its Maclaurin series for $x = 0.1$,

$$\frac{dy}{dx} = \frac{1}{2}(1+x)y^2, \quad y(0) = 1.$$

Compare the value with the exact solution.

Exercises

19. Employ Taylor's method to obtain approximate value of y at $x = 0.2$ for the differential equation

$$\frac{dy}{dx} = 2y + 3e^x, \quad y(0) = 0.$$

Compare the numerical solution obtained with the exact solution.

20. Solve by Taylor series method of the equation

$$\frac{dy}{dx} = \frac{x^3 + xy^2}{e^x}$$

$y(0) = 1$ for y at $x = 0.1, 0.2$ and $x = 0.3$.

21. Solve by Taylor's series method the equation

$$\frac{dy}{dx} = \log(xy)$$

for $y(1.1)$ and $y(1.2)$, given that $y(1) = 2$.

22. From the Taylor series for $y(x)$, find $y(0.1)$ correct to 4 decimal places if $y(x)$ satisfies $y' = x - y^2$ and $y(0) = 1$.

23. Using Taylor's method solve

$$\frac{dy}{dx} = 1 + xy$$

with $y(0) = 2$. Find y at $x = 0.1,\ 0.2,\ 0.3$.

24. Using Taylor's method find $y(0.1)$ correct to 3 decimal places from

$$\frac{dy}{dx} + 2xy = 1$$

with $y(0) = 0$.

Solving Simultaneous First Order Differential Equations : Taylor's Series Method

Consider the simultaneous differential equations of the type

$$\frac{dy}{dx} = f(x, y, z) \tag{7}$$

and

$$\frac{dz}{dx} = g(x, y, z) \tag{8}$$

with initial conditions

$$y(x_0) = y_0 \quad \text{and} \quad z(x_0) = z_0$$

can be solved by Taylor's series method.

Solving Simultaneous First Order Differential Equations : Taylor's Series Method

Let h be the interval of differencing

$$y_1 = y(x_0 + h) \quad \text{and} \quad z_1 = z_0 + h.$$

The Taylor's algorithm for (7) and (8) are given by

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots \tag{9}$$

$$z_1 = z_0 + hz_0' + \frac{h^2}{2!}z_0'' + \frac{h^3}{3!}z_0''' + \cdots . \tag{10}$$

Differentiating (7) and (8) successively, we get y'', z'' etc. Hence, the values of $y_0', y_0'', y_0''', \ldots,$ and $z_0', z_0'', z_0''', \ldots$ can be found.

Solving Simultaneous First Order Differential Equations : Taylor's Series Method

Substituting these values in (9) and (10) we get the first approximation y_1 and z_1. Similarly for the second approximation, the Taylor's algorithms are

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \cdots \quad (11)$$

$$z_2 = z_1 + hz_1' + \frac{h^2}{2!}z_1'' + \frac{h^3}{3!}z_1''' + \cdots . \quad (12)$$

Using the first approximate values y_1 and z_1, we can calculate y_1', y_1'' etc. and z_1', z_1'' etc.

Substituting these values in (11) and (12) we get y_2 and z_2. Proceeding like this we can calculate the higher approximations of y and z step by step.

25. Solve the initial value problem involve two independent functions $x(t)$ and $y(t)$ using Taylor series method,

$$\frac{dx}{dt} = ty + 1, \frac{dy}{dt} = -tx$$

$x = 0, y = 1$ when $t = 0$. Evaluate x and y at $t = 0.1$.

26. Find an approximate series solution of the simultaneous equations

$$\frac{dx}{dt} = xy + 2t, \quad \frac{dy}{dt} = 2ty + x$$

subject to the initial conditions $x = 1,\ y = -1$ when $t = 0$.

Solving Second / Higher Order Differential Equations : Picard's Method

Consider a second order differential equation

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right).$$

Let $z = \frac{dy}{dx}$. Hence the given equation can be reduced to 2 first order simultaneous differential equations

$$\frac{dy}{dx} = z$$

and

$$\frac{dz}{dx} = f(x, y, z).$$

27. Using Taylor's series method obtain the values of y at $x = 0.1$ to 4 significant figures, if y satisfies the equation

$$\frac{d^2y}{dx^2} = -xy$$

given that $\frac{dy}{dx} = 0.5$ and $y = 1$ when $x = 0$.

28. Find the first 6 terms of the power series method solution of

$$y'' + xy' - 2y = 0$$

if $y(1) = 0, \; y'(1) = 1.$

29. Find the value of $y(1.1)$ and $y(1.2)$ from

$$y'' + y^2 y' = x^3 0$$

with the conditions $y(1) = 1$ and $y'(1) = 1$, using Taylor's series method.

30. Using Taylor's series method, obtain the values of y at $x = 0.2$ correct to 4 decimal places, if y satisfies the equation

$$\frac{d^2y}{dx^2} = xy$$

given that $y = 1$ and $\frac{dy}{dx} = 1$ at $x = 0$.

31. Given the differential equation

$$y'' - xy' - y = 0$$

with the conditions $y(0) = 1$ and $y'(0) = 0$, use Taylor's series method to determine the value of $y(0.1)$.

32. Using Taylor's series method, find the values of x and y for $t = 0.4$ satisfying the differentiatl equations

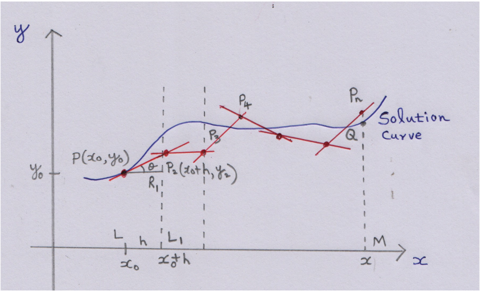
$$\frac{dx}{dt} = x + y + t, \quad \frac{d^2y}{dt^2} = x - t$$

with initial conditions $x = 0,\ y = 1, \frac{dy}{dt} = 1$ at $t = 0$.

Euler's Method

Consider the equation $\frac{dy}{dx} = f(x, y)$ given $y(x_0) = y_0$.

For a given point $P(x_0, y_0)$ in the (unknown) solution curve, we are going to find the ordinate of x, possibly near to the solution curve.



The unknown solution curve is in blue, and its polygonal approximation is in red. Let us divide LM into n subintervals each of width h, the value of h is quite small.

Euler's Method

In the interval LL_1, we approximate the curve by the tangent at P.

If the ordinate through L_1 meets this tangent in $P_1(x_0 + h, y_1)$, then

$$\begin{aligned} y_1 &= L_1 P_1 = LP + R_1 P_1 \\ &= y_0 + PR_1 \tan\theta \\ &= y_0 + h\left(\frac{dy}{dx}\right)_P \\ &= y_0 + hf(x_0, y_0). \end{aligned}$$

Euler's Method

Let P_1Q_1 be the solution curve of the given initial value problem through P_1 and let its tangent at P_1 meet the ordinate through L_2 in $P_2(x_0 + 2h, y_2)$.

Then

$$y_2 = y_1 + hf(x_0 + h, y_1).$$

Repeating this process n times, we finally reach on an approximation MP_n of MQ given by

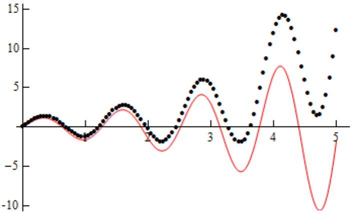
$$y_n = y_{n-1} + hf(x_0 + (n-1)h, y_{n-1}).$$

This is Euler's method of finding an approximate solution of the given differential equation

$$\frac{dy}{dx} = f(x, y) \qquad \text{given} \quad y(x_0) = y_0.$$

Drawbacks of Euler's Method

In Euler's method, we approximate the solution curve by the tangent in each interval, i.e., by a sequence of **points** (or by short lines).

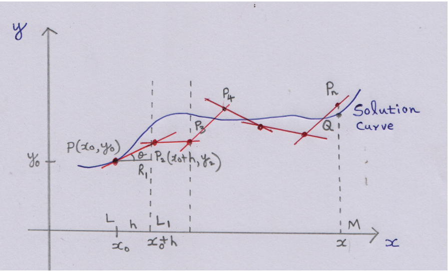


Unless h is small, the error is bound to be quite significant. **This sequence of lines may also deviate considerably from the solution curve.**

As such, the method is slow and hence there is a modification of this method, which will be discussed later.

Drawbacks of Euler's Method

An another drawback lies in the fact that if $\frac{dy}{dx}$ changes rapidly over an interval, then its value at the beginning of the interval may give a poor approximation as compared to its average value over the interval and thus the value of y calculated from Euler's method may be in much error from its true value.



These errors accumulate in the succeeding intervals and the value of y becomes much erroroneous ultimately. Thus the method is completely useless.

Exercises

33. Using Euler's method, find an approximate value of y corresponding to $x = 1$, given that

$$\frac{dy}{dx} = x + y \qquad \text{and} \qquad y(0) = 1.$$

34. Solve for $y(0.4)$ by Euler's method, $y' = -y$ with the condition $y(0) = 1$.

35. Given

$$\frac{dy}{dx} = \frac{y - x}{y + x}$$

with initial condition $y = 1$ at $x = 0$; find y for $x = 0.1$ by Euler's method.

36. Using Euler's method, compute y for $x = 0.1$ and 0.2 choosing $h = 0.1$ from $y' = y - 2x/y$, $y(0) = 1$.

37. Determine the numerical solution at $x = 0.8$ for

$$\frac{dy}{dx} = \sqrt{x + y}$$

given $y(0.4) = 0.41$. Assume $h = 0.2$.

38. Solve

$$\frac{dy}{dt} + y^2 = 0$$

given $y = 1$ at $t = 0$. Find approximately the value of y for $t = 1$ by Euler's method in five steps.

39. Solve the equation

$$\frac{dy}{dx} = 1 - y^2$$

with $x = 0, y = 1$, using Euler's algorithm and tabulate the solutions of $x = 0.1, 0.2, 0.3, 0.4$.

Modified Euler's Method

In the Euler's method, the curve of solution in the interval LL_1 is approximated by the tangent at P such that at P_1, we have

$$y_1 = y_0 + hf(x_0, y_0).$$

Then the slope of the curve of solution through P_1,

$$\left(\frac{dy}{dx}\right)_{P_1} = f(x_0 + h, y_1)$$

is computed and the tangent at P_1 to P_1Q_1 is drawn meeting the ordinate through L_2 in $P(x_0 + 2h, y_2)$.

Modified Euler's Method

Now we find a better approximation $y_1^{(1)}$ of $y(x_0 + h)$ by taking the slope of the curve as the mean of the slopes of the tangents at P and P_1.

That is,

$$y_1^{(1)} = y_0 + \frac{h}{2}\Big[f(x_0, y_0) + f(x_0 + h, y_1)\Big].$$

As the slope of the tangent at P_1 is not known, we calculate y_1 by Euler's method to calculate the **first modified value** $y_1^{(1)}$.

Modified Euler's Method

We can find a still better value $y_1^{(2)}$ corresponding to L_1 as

$$y_1^{(2)} = y_0 + \frac{h}{2}\Big[f(x_0, y_0) + f(x_0 + h, y_1^{(1)})\Big].$$

We repeat this step, till two consecutive values of y agree. This is then taken as the starting point for the next interval $L_1 L_2$.

Once y_1 is obtained to desired degree of accuracy, y corresponding to L_2 is found from Euler's method, and a better approximation $y_2^{(1)}$ is obtained

$$y_2^{(1)} = y_1 + \frac{h}{2}\Big[f(x_0 + h, y_1) + f(x_0 + 2h, y_2)\Big].$$

We repeat this step until y_2 becomes stationary. Then we calculate y_3. This is the **modified Euler's method** which gives great improvement in accuracy over the original method.

Exercises

40. Using modified Euler's method, find an approximate value of y when $x = 0.3$ given that

$$\frac{dy}{dx} = x + y$$

and $y = 1$ when $x = 0$.

41. Using modified Euler's method, find $y(0.2)$ and $y(0.4)$ given

$$\frac{dy}{dx} = y + e^x \qquad y(0) = 0.$$

42. Solve the following differential equation by Euler's modified method,

$$\frac{dy}{dx} = \log(x + y) \qquad y(0) = 2.$$

Exercises

43. Using modified Euler's method, obtain a solution of the equation

$$\frac{dy}{dx} = x + |\sqrt{y}|$$

with the initial condition $y = 1$ at $x = 0$, for the range $0 \leq x \leq 0.6$ in steps of 0.2.

44. Using modified Euler's method, to compute $y(0.2)$ with $h = 0.1$ from

$$\frac{dy}{dx} = y - \frac{2x}{y}, \qquad y(0) = 1.$$

45. Solve by modified Euler's method,

$$\frac{dy}{dx} = x + y^2, \qquad y(0) = 1.$$

Exercises

46. Using modified Euler's method, solve

$$\frac{dy}{dx} = x^2 + y, \qquad y(0) = 0.94$$

for $x = 0.1$.

47. Use modified Euler's method to find an approximate solution at the point 0.2 of the problem

$$y' = -y \qquad y(0) = 1$$

with step length $h = 0.2$.

48. Solve

$$\frac{dy}{dx} = 1 - y, \qquad y(0) = 0$$

in the range $0 \leq x \leq 0.2$ by taking $h = 0.1$.
Compare the result with the results of the exact solution.

49. Use Euler's method solve

$$\frac{dy}{dx} = 1 + xy \qquad y(0) = 2.$$

Find $y(0.1), y(0.2)$ and $y(0.3)$. Also find the values by modified Euler's method

50. Given

$$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}, \qquad y(1) = 1.$$

Evaluate $y(1.3)$ by modified Euler's method.

51. Solve the simultaneous equations by the modified Euler's method with $h = 0.1$, $y' = z$, $z' = y + xz$, $y(0) = 1, z(0) = 0$.

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