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Computational Linear Algebra - MA 703
Problem Sheet 3

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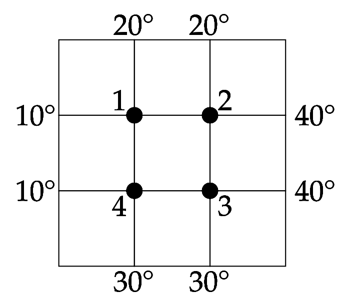
1. Assume the plate shown in the figure represents a cross section of a metal beam with negligible heat flow in the direction perpendicular to the plate. Let T_1, T_2, T_3, T_4 denote the temperature at the four interior nodes 1, 2, 3 and 4 of the mesh in the figure. The temperature at a node is equal to the average of the four nearest nodes (left, right, above and below). Find T_1, T_2,T_3 and T_4.



2. Let P_3 denote the set of cubic polynomials. For each real x, let S_x denote the cubic polynomials having x as a root. For what values of x, S_x is a subspace of P_3?

3. Give two 2×2 matrices A and B such that $rank(A) = rank(B)$, but $rank(A^2) \neq rank(B^2)$.

4. Define linearly independent set. Give an example in $\mathbb{R}$.

5. Construct a 4×4 matrix whose null space and range space are same. How about the same for 5×5 and 6×6 matrices?

6. True or false : give a specific counter example when false.
   Let $S_{m\times m}$ and $T_{n\times n}$ be two matrices. Then rank(S) and rank(T) are equal only when $m = n$.

7. Find two 2×2 matrices K and L such that
   (a) $Kx = 0$ has only the trivial solution.
   (b) $Lx = 0$ has a non-trivial solution.

8. Write down a basis for the vector space of all 3×3 real symmetric matrices.

9. For what values of a, the matrix $\begin{pmatrix} -1 & a \\ 0 & -1 \end{pmatrix}$ is not similar to the matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$?

10. State the rank-nullity theorem and verify it for the matrix $A = \begin{pmatrix} 3 & 1 & 4 \\ 7 & 2 & 8 \end{pmatrix}$.

11. Find all solutions of $Ax = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, where $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$.
    (a) Express the solution in the form $x = x_{\text{homogeneous}} + x_{\text{particular}}$.
    (b) Find a basis for the nullspace of A.
    (c) Does there exist a vector $b \in \mathbb{R}^2$ such that $Ax = b$ has no solution?

12. No two subspaces of a vector space are disjoint. Why?

13. For any matrix A, show that the column space, $C(A) = \{0\}$ if and only if $A = 0$.

14. What is the dimension of the vector space of 2×2 matrices with real entries? Write down a basis.

15. Find the inverse of $\begin{pmatrix} 1 & -4 & 4 \\ 2 & 2 & 0 \\ -2 & 3 & 3 \end{pmatrix}$ by Gauss-Jordan method.

16. Find the inverse of the matrix B by partitioning where $B = \left[\begin{array}{c|cc} 1 & 2 & 3 \\ \hline 1 & 3 & 2 \\ 1 & 1 & 5 \end{array}\right]$.

17. Find the rank of the following matrix by reducing it to an echelon matrix:

$$\begin{bmatrix} 2 & 1 & 4 & 0 & 6 \\ 3 & 5 & 9 & 1 & 0 \\ 7 & 7 & 3 & 2 & 8 \\ -9 & -8 & 7 & -3 & -10 \\ 4 & 2 & -20 & 2 & 4 \end{bmatrix}$$

18. If $v_1, v_2, \dots, v_n$ is a basis of V over $\mathbb{R}$ and if $w_1, w_2, \dots, w_m$ in V are linearly independent over $\mathbb{R}$, then show that $m \leq n$.

19. By stating all the required results, show that any two bases of $\mathbb{R}^n$ have the same number of basis vectors.

20. Show that $W = \{(x, 2x + y, y - z, z) : x, y, z \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^4$ and find a basis for W.

21. Let $A = \begin{pmatrix} 1 & 0 & 0 & 4 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & t \end{pmatrix}$ and $b = \begin{pmatrix} 2 \\ 0 \\ 0 \\ s \end{pmatrix}$.
    (a) For any real number t, and any real number s, find the complete solution to the equation $Ax = b$. Note that the complete solution depends on t and s.
    (b) For which t, are the columns of the matrix A linearly dependent?
    (c) Consider b and the first three columns of A. For which s, are these linearly dependent?

22. Finding inverse of $B = \begin{pmatrix} 1 & 3 & 0 \\ 2 & -2 & 1 \\ -4 & 1 & -1 \end{pmatrix}$ by elementary row operations.
    The augmented matrix $[N\ c]$ is row reduced to $[I\ d]$. What is the relation between N, c and d ?

23. Find the inverse of the above matrix B by the method of partitioning.

24. Let $B = \begin{pmatrix} 3 & -5 & 1 \\ 0 & 0 & 1 \\ 3 & -7 & 8 \end{pmatrix}$. Find the matrices A and C such that ABC is in row reduced echelon form.

25. Give a basis for each of the four fundamental subspaces associated to the following matrix $\begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}$.

26. Suppose $A = \begin{pmatrix} 1 & 2 & 1 & b \\ 2 & a & 1 & 8 \\ (row & 3 & of & A) \end{pmatrix}$ has reduced echelon form $R = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.
    (a) What can you say about row 3 of A?
    (b) What are the numbers a and b?
    (c) Describe the nullspace of A.