Results from Calculus

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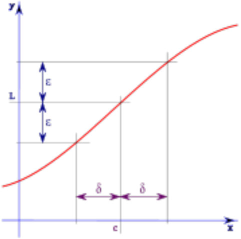
Limit of a Function

A function f defined on a set X of real numbers has the **limit** L at c, written

$$\lim_{x \to c} f(x) = L,$$

if, given any real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that $|f(x) - L| < \varepsilon$, whenever $x \in X$ and $|x - c| < \delta$.



Continuous Function

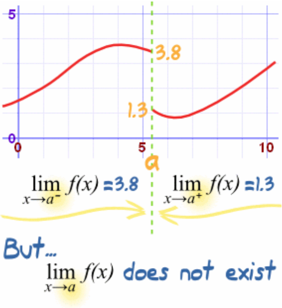
Let f be a function defined on a set X of real numbers and $c \in X$. Then f is **continuous** at c if

$$\lim_{x \to c} f(x) = f(c).$$

The function f is continuous on the set X if it is continuous at each number in X.

The set of all continuous functions on X is denoted by $C(X)$.

Example of a Discontinuous Function

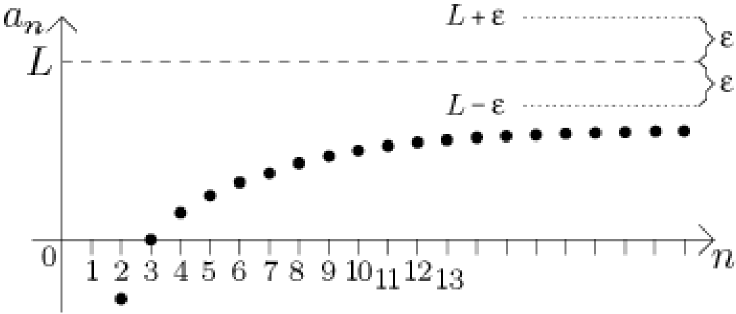


Convergent Sequence

Let $(x_n)_{n=1}^{\infty}$ be an infinite sequence of real or complex numbers. The sequence $(x_n)_{n=1}^{\infty}$ has the **limit** L (**converges to** L) if, for any $\varepsilon > 0$, there exists a positive number $N(\varepsilon)$ such that $|x_n - L| < \varepsilon$, whenever $n > N(\varepsilon)$.

The notation $\lim_{n\to\infty} x_n = L$, or $x_n \to L$ as $n \to \infty$, means that the sequence $(x_n)_{n=1}^{\infty}$ converges to L.

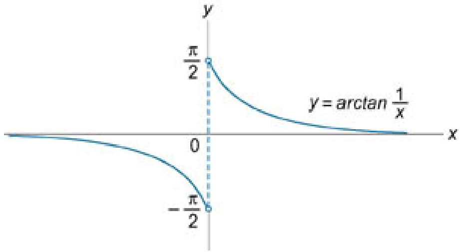
Graph of a Convergent Sequence



Continuity Theorem for Convergent Sequences

If f is a function defined on a set X of real numbers and $L \in X$, then the following statements are equivalent:

1. f is continuous at L;
2. If $(x_n)_{n=1}^{\infty}$ is any sequence in X converging to L, then $\lim_{x\to\infty} f(x_n) = f(L)$.



All the functions we will consider when discussing numerical methods will be assumed to be continuous since this is a minimal requirement for predictable behavior. Functions that are not continuous can skip over points of interest, which can cause difficulties when attempting to approximate a solution to a problem.

More sophisticated assumptions about a function generally lead to better approximation results. For example, a function with a smooth graph will normally behave more predictably than one with numerous jagged features. The smoothness condition relies on the concept of the derivative.

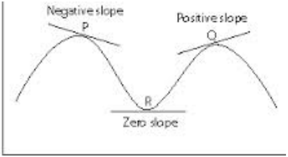
Differentiable Functions

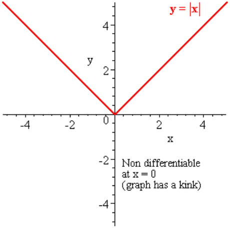
Let f be a function defined in an open interval containing x_0. The function f is **differentiable** at x_0 if

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

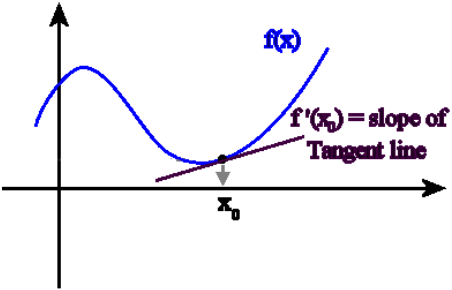
exists. The number $f'(x_0)$ is called the **derivative** of f at x_0. A function that has a derivative at each number in a set X is **differentiable** on X.

Example





The derivative of f at x_0 is the slope of the tangent line to the graph of f at $(x_0, f(x_0))$, as shown in the following figure.



Theorem

If the function f is differentiable at x_0, then f is continuous at x_0.

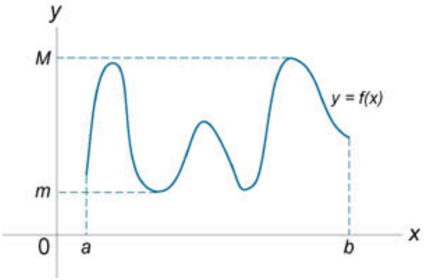
The set of all functions that have n continuous derivatives on X is denoted by $C^n(X)$, and the set of functions that have derivatives of all orders on X is denoted by $C^\infty(X)$.

Polynomial, rational, trigonometric, exponential, and logarithmic functions are in $C^\infty(X)$, where X consists of all numbers for which the functions are defined.

The next certain mathematical results are of fundamental importance is deriving methods for error estimation. We state them, without proof.

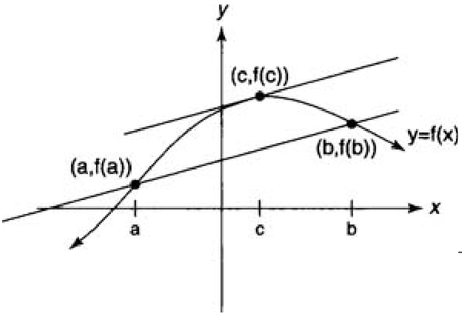
Extreme Value Theorem

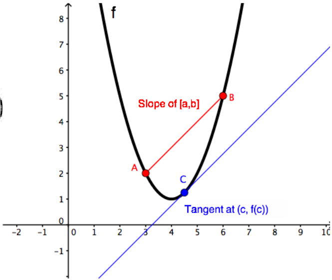
Let f be a continuous function on the interval $[a, b]$. Let $M = \sup_{a \leq x \leq b} f(x)$ and $M = \inf_{a \leq x \leq b} f(x)$. Then there are points c_1, c_2 in $[a, b]$ such that $m = f(c_1) \leq f(x) \leq f(c_2) = M$, for all $x \in [a, b]$. In addition, if f is differentiable on (a, b), then the numbers c_1 and c_2 occur either at the endpoints of $[a, b]$ or where f' is zero.



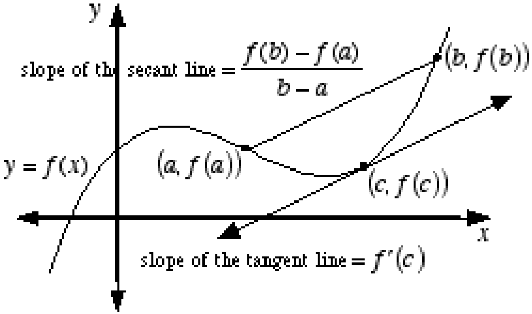
Mean Value Theorem

Let f be continuous on $[a, b]$ and differentiable in (a, b). Then there is at least on point $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.



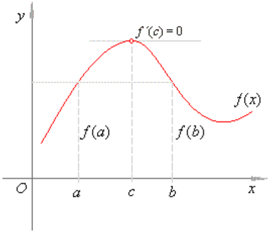


The tangent line and the secant line are parallel, so they have the same slope.



Theorem (Rolle's Theorem)

Let f be continuous on $[a, b]$ and differentiable in (a, b). If $f(a) = f(b)$, then there is at least on point $c \in (a, b)$ such that $f'(c) = 0$.



Theorem (Generalized Rolle's Theorem)

Let f be continuous on $[a, b]$ and n times differentiable in (a, b). If $f(x)$ is zero at the $n + 1$ distinct numbers $c_0, c_1, \ldots, c_n$ in $[a, b]$, then a number c in (a, b) exists with $f^{(n)}(c) = 0$.

Integral Mean Value Theorem

Let $f(x)$ be continuous on $[a, b]$, and let $w(x)$ be nonnegative and integrable on $[a, b]$. Then

$$\int_a^b f(x)w(x)dx = f(\xi)\int_a^b w(x)dx,$$

for some $\xi \in [a, b]$.

Let f be a continuous function on $[a, b]$ and K is any number between $f(a)$ and $f(b)$, then there exists a number c in (a, b) for which $f(c) = K$.

The Intermediate Value Theorem is used to determine when solutions to certain problems exist.