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Problem Sheet 4

1. Prove or disprove: $\text{Sp}(A) \cap \text{Sp}(B) \neq \{0\} \implies A \cap B \neq \emptyset$.
2. True or False : If $A \subseteq B$ and $\text{Sp}(A) \supseteq B$, then $\text{Sp}(A) = \text{Sp}(B)$.
3. Let X be the set of all positive integers. In the vector space $\mathbb{R}^X$ of all real-valued functions on X, what is the span of the set $A = \{f_i : i \geq 1\}$, where f_i is the function in $\mathbb{R}^X$ taking value 1 at $x = i$ and 0 elsewhere? Show that if $f \in \texttt{Sp}(A)$ then the range of f is finite but the converse is not true.
4. True or false : In the vector spce $\mathbb{R}$ over the field $\mathbb{Q}$, the sets $\{1, \sqrt{2}\}$, $\{\sqrt{2}, \sqrt{3}, \sqrt{6}\}$ and $\{\sqrt{2}, \sqrt{3}, \sqrt{12}\}$ are linearly independent.
5. If x and y are linearly independent show that $x + \alpha y$ and $x + \beta y$ are linearly independent whenever $\alpha \neq \beta$.
6. Let $\texttt{Sp}(A) = S$. Then show that no proper subset of A generates S iff A is linearly independent.
7. For what values of α are the vectors $(0, 1, \alpha), (\alpha, 1, 0)$ and $(1, \alpha, 1)$ in $\mathbb{R}^3$ linearly independent.
8. Let $S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Compute the least positive integer k such that S^k is the zero matrix.
9. Find E^2, E^8 and E^{-1} if $E = \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}$.
10. Let $P_{n\times n}$ be any permutation matrix. Prove that $P^m_{n\times n} = I_{n\times n}$ for some m.
11. In each of the following, find precisely which axioms in the definition of a vector space are violated. Take $V = \mathbb{R}^2$ and $F = \mathbb{R}$ throughout
    (a) $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, 0), \quad \alpha(x_1, x_2) = (\alpha x_1, 0)$
    (b) $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2), \quad \alpha(x_1, x_2) = (\alpha x_1, 0)$
    (c) $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2), \quad \alpha(x_1, x_2) = (\alpha x_1, 2\alpha x_2)$
    (d) $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2), \quad \alpha(x_1, x_2) = (\alpha + x_1, \alpha + x_2)$.
12. True or False : The set of all positive real numbers forms a vector space over $\mathbb{R}$ if the sum of x and y is defined to be the usual product xy and α times x is defined to be x^α.
13. Let V be a vector space. On $V \times V$, define +, and . as follows:
$$\begin{aligned} (x_1, y_1) + (x_2, y_2) &= (x_1 + y_1, x_2 + y_2) \\ \alpha(x, y) &= (\alpha x, \alpha y), \alpha \in \mathbb{R}, x, y \in V. \end{aligned}$$
Is $V \times V$ a vector space? If not, write down the conditions (axioms) which are violated.
14. Let $X := \{*\}$ be a singleton set and let V be a vector space. Let $W = \{*\} \times V$. Can we turn W into a vector space as follows?
$$\begin{aligned} (*, x_1) + (*, x_2) &= (*, x_1 + x_2), x_1, x_2 \in V \\ \alpha(*, x) &= (*, \alpha x), \alpha \in \mathbb{R}, x \in V. \end{aligned}$$