Rank-Factorization of a Matrix

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Every non-null matrix can be written as a product of two full rank matrices. Martrices which are of full rank (either full row rank or full column rank) have several nice properties.

Definition

Let A be a $m \times n$ matrix. Then the **column space** *of A is $\mathcal{C}(A)$ is*

$$\mathcal{C}(A) := \{Ax : x \in F^n\}$$

and the **row space** *of A is*

$$\mathcal{R}(A) := \{y^T A : y \in F^m\}.$$

- We call $dim(\mathcal{R}(A))$ the **row rank** of A and $dim(\mathcal{C}(A))$ the **column rank** of A.
- We refer to a basis of $\mathcal{C}(A)$ consisting of columns of A as a **column basis**. A **row basis** is defined similarly.

Matrix Multiplication.

Notation. A_{i*} denotes the i-th row of A and A_{*j} denotes the j-th column of A.

Let A, B, C be matrices of orders $m \times n$, $n \times p$, and $p \times q$ respectively. Then

1. $(AB)_{ij} = A_{i*}B_{*j}$,
2. $(AB)_{i*} = A_{i*}B$,
3. $(AB)_{*j} = AB_{*j}$,
4. $(ABC)_{ij} = A_{i*}BC_{*j}$.
5. For any $m \times n$ matrix A, we have $A_{i*} = e_i^T A$ and $A_{*j} = Ae_j$.

If A has column rank r, then

- any r linearly independent columns of A form a basis for $\mathcal{C}(A)$,
- every maximal linearly independent set of columns of A contains exactly r vectors,
- any r columns of A which generate $\mathcal{C}(A)$ form a basis of $\mathcal{C}(A)$.

Theorem

For any matrix A, the row rank of A equals the column rank of A.

Proof. Let A be a $m \times n$ matrix with row rank r and column rank s. If $A = 0$, then $\mathcal{R}(A) = \{0\}$ and $\mathcal{C}(A) = \{0\}$, so $r = s = 0$ and we are done.

Let $B = [x_1 : x_2 : \cdots : x_s]$ be an $m \times s$ matrix whose columns form a basis of $\mathcal{C}(A)$. Then for each $j = 1, 2, \ldots, n$, each column of A, A_{*j} is a linear combination of the columns of B, so there exists an $s \times 1$ vector y_j such that $A_{*j} = By_j$.

Now

$$\begin{aligned} A &= [A_{*1} : \cdots : A_{*n}] = [By_1 : \cdots : By_n] \\ &= B[y_1 : \cdots : y_n] = BC \end{aligned}$$

where $C = [y_1 : \cdots : y_n]$. Note that C is of size $s \times n$.

Since $A = BC$, $A_{i*} = B_{i*}.C$, and each row of A is a linear combination of the rows of C. Thus $\mathcal{R}(A) \subseteq \mathcal{R}(C)$.

Taking dimensions, we get $r \leq row\ rank(C)$.

As C has only s rows, $row\ rank(C) \leq s$. Hence $r \leq s$.

Interchanging the roles of row rank and the column rank. Let $C = [y_1 : \cdots : y_r]^T$ be an $r \times n$ matrix whose rows form a basis of $\mathcal{R}(A)$. Then for each $i = 1, 2, \ldots, n$, each row of A, A_{i*} is a linear combination of the rows of C, so there exists an $r \times 1$ vector x_i such that $A_{i*} = x_i C$.

$$\begin{aligned} A &= [A_{1*} : \cdots : A_{n*}]^T = [x_1 C : \cdots : x_n C]^T \\ &= [x_1 : \cdots : x_n]^T C = BC \end{aligned}$$

where $B = [x_1 : \cdots : x_n]^T$. Note that C is of size $s \times n$.

Since $A = BC$, $A_{*j} = B.C_{*j}$, and each column of A is a linear combination of the columns of C. Thus $\mathcal{C}(A) \subseteq \mathcal{C}(B)$.

Taking dimensions, we get $s \leq column\ rank(B)$. As B has only r columns, $column\ rank(B) \leq r$. Hence $s \leq r$. Thus $r = s$.

Definition

*The **rank** of a matrix A is the common value of the row rank of A and the column rank of A and is denoted by $\rho(A)$.*

- The rank of an $m \times n$ matrix obviously lies between 0 and $\min(m, n)$.
- Conversely, given any non-negative integer $r \leq \min(m, n)$, there exists an $m \times n$ matrix A with rank r.
- Let A be a $m \times n$ matrix of rank r and B a submatrix of A. By considering row rank (column rank) if B is obtained from A by omitting only some rows (columns). Any submatrix can be obtained by omitting some rows and then some columns. Then $\rho(B) \leq \rho(A)$.

Definition

An $m \times n$ matrix A is said to be of **full row rank** *if its rows are linearly independent, that is, it its rank is m. Similarly A is said to be of* **full column rank** *if its columns are linearly independent.*

A **left inverse** *of a matrix A is any matrix B such that $BA = I$. A* **right inverse** *of A is any matrix C such that $AC = I$.*

A matrix B is said to be an **inverse** *of A if it is both a left inverse and a right inverse of A.*

Theorem

Let A be a $m \times n$ matrix over F. Then the following statements are equivalent.

1. *A has a right inverse.*
2. **Right cancellation law:** $XA = YA \Rightarrow X = Y$.
3. $XA = 0 \Rightarrow X = 0$.
4. *A is of full row rank.*
5. **The linear transformation** $f : x \mapsto Ax$ **is onto:** $\mathcal{C}(A) = F^m$.

Question: If A has a right inverse, how many right inverses does A have ?

Theorem

Let A be a $m \times n$ matrix over F. Then the following statements are equivalent.

1. *A has a left inverse.*
2. **Left cancellation law:** $AX = AY \Rightarrow X = Y$.
3. $AX = 0 \Rightarrow X = 0$.
4. *A is of full column rank.*
5. **The linear transformation** $f : x \mapsto Ax$ **is one-to-one:** $\mathcal{R}(A) = F^n$.

- A matrix B is a left inverse of a matrix A iff B^T is a right inverse of A^T.
- If B and C are left inverses of A, then $\alpha B + (1 - \alpha)C$ is also a left inverse of A.

If a matrix A has a left inverse B and a right inverse C, then

- A is square,
- $B = C$,
- A has a unique left inverse, a unique right inverse and a unique inverse.

If a matrix A has an inverse, then A^{-1} is unique, A is square and $AA^{-1} = A^{-1}A = I$.

Theorem

Let A be a square matrix of order n. Then the following statements are equivalent:

1. *A has a right inverse*
2. *rank of A is n*
3. *A has a left inverse*
4. *A has an inverse.*

Definition

A square matrix A is said to be **non-singular** *if it has an inverse. A square matrix which does not possess an inverse is said to be* **singular**.

1. AB is non-singular iff both A and B are non-singular.
2. If A is non-singular and k is a positive integer, then A^k is non-singular and its inverse is $(A^{-1})^k$.
3. The sum of two non-singular matrices need not be non-singular.
4. Let P be a permutation matrix. Then P is non-singular and $P^{-1} = P^T$.
5. If P is a permutation matrix obtained from I by interchanging two rows, $P^{-1} = P$.

Let A be a non-singular matrix whose inverse if of interest. Sometimes it happens that it is easier to compute the inverse of a matrix B obtained from A by permuting the columns (or rows). **How do we get the inverse of A from that of B ?**

Theorem

Let B be obtained from a non-singular matrix A by **permuting the columns** *so that j-th column of B is the i_j-th column of A for $j = 1, 2, \ldots, n$, where $(i_1, i_2, \ldots, i_n)$ is a permutation of $(1, 2, \ldots, n)$. Then A^{-1} can be obtained from B^{-1}* **by permuting the rows** *thus :*
the i_j-th row of A^{-1} is the j-th row of B^{-1}.

Definition

An $n \times n$ complex (or real) matrix A is said to be **strictly diagonally dominated** *if for each $i = 1, 2, \ldots, n$,*

$$|a_{ij}| > \sum_{j=1, j \neq i}^{n} |a_{ij}|.$$

Theorem

Every strictly diagonally dominated matrix has an inverse.

Definition

Let A be a $m \times n$ matrix with rank $r \geq 1$. Then (P, Q) is said to be a **rank-factorization** *of A if P is of order $m \times r$, Q is of order $r \times n$ and $A = PQ$.*

Theorem

Every non-null matrix has a rank-factorization.

Proof. Let A be a $m \times n$ matrix with rank r.

Let $B = [x_1 : x_2 : \cdots : x_r]$ be an $m \times r$ matrix whose columns form a basis of $\mathcal{C}(A)$. Then for each $j = 1, 2, \ldots, n$, each column of A, A_{*j} is a linear combination of the columns of B, so there exists an $r \times 1$ vector y_j such that $A_{*j} = By_j$.

Now

$$\begin{aligned} A &= [A_{*1} : \cdots : A_{*n}] \\ &= [By_1 : \cdots : By_n] \\ &= B[y_1 : \cdots : y_n] \\ &= BC \end{aligned}$$

where $C = [y_1 : \cdots : y_n]$.

- **A null matrix cannot have a rank-factorization** since there cannot be a matrix with 0 rows.
- **Rank-factorization of a matrix is not unique.** The choice of the matrix B is not unique because the columns of B are coming from the column basis of A.
- If (B, C) is a rank-factorization of A, then (C^T, B^T) is a rank-factorization of A^T.

When a factorization is a rank-factorization?

Theorem

Let $A = PQ$ where P is a $m \times k$ matrix and Q a $k \times n$ matrix. Then the rank of A is at most k.
Moreover, the following are equivalent:

- *the rank of A is k,*
- *(P, Q) is a rank-factorization of A,*
- *P is of full column rank and Q is of full row rank,*
- *the columns of P form a basis of $\mathcal{C}(A)$,*
- *the row of Q form a basis of $\mathcal{R}(A)$.*

Corollary

If (P, Q) is a rank-factorization of A then $\mathcal{C}(P) = \mathcal{C}(A)$, $\mathcal{R}(Q) = \mathcal{R}(A)$ and $\mathcal{N}(Q) = \mathcal{N}(B)$.

Theorem

If $A = A^2$, rank of A equals trace of A.

Proof. The result is trivial if the rank r of A is 0, so let $r \geq 1$.

Let (P, Q) be a rank-factorization of A. Then $PQPQ = PQ = PI_rQ$.

Since P is of full column rank and Q is of full row rank, left and cancellation laws are applied, we get $PA = I_r$.

Hence rank of $A = r = tr(I_r) = tr(QP) = tr(PQ) = tr(A)$.

Finding a rank-factorization of a matrix A of rank r is easy when A is represented in the following nice form.

Theorem

Let A be an $m \times n$ matrix of rank $r \geq 1$. Then there exist permutation matrices P and Q such that

$$A = P \begin{pmatrix} B & BC \\ DB & DBC \end{pmatrix} Q$$

where B is non-singular matrix of order r and, C and D are some matrices of orders $r \times (n-r)$ and $(m-r) \times r$ respectively.

When a matrix A in the above form, can be factorized as $A = P_1 Q_1$ where

$$P_1 = P \begin{pmatrix} B \\ DB \end{pmatrix} \text{ and } Q_1 = \begin{pmatrix} I_r & : & C \end{pmatrix} Q.$$

Since P_1 is of order $m \times r$, it follows that (P_1, Q_1) is a rank-factorization of A.

References

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