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MA939 Functional Analysis
Problem Sheet - 4

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Answer **ALL** questions.

1. Let I be the interval $[0,1]$ with the Lebesgue measure μ. Let X be the collection of all bounded measurable functions on I, and for $x \in X$ let
$$\|x\| := \sup_{s \in X} |x(s)|.$$
Prove that X is a Banach space.
2. Let X be the space as defined above. For $1 \leq p < \infty$ and $x \in X$, whether X is a Banach space with respect to $\|x\|_p = \left(\int_0^1 |x(s)|^p \, ds\right)^{1/p}$ or not.
3. Prove that $C[a,b]$ is a proper closed subspace of $L_\infty[a,b]$.
4. Prove that for a fixed point $t_0 \in [a,b]$, the map $x \mapsto |x(t_0)|$, $x \in C[a,b]$, is a seminorm on $C[a,b]$.
5. Let $1 \leq p < \infty$. Prove that the maps $x \mapsto \sum_{j=0}^{k} \|x^{(j)}\|_p$, $x \mapsto \max_{0 \leq j \leq k} \|x^{(j)}\|_p$ are norms on $C^k[a,b]$. Moreover, $C^\infty[a,b]$ is not Banach in the induced norm of $C^k[a,b]$.
6. Prove that $C^k[a,b]$ is not Banach with respect to any norm $\|.\|_p$ for $1 \leq p \leq \infty$ but it is Banach with respect to $\sum_{j=0}^{k} \|x^{(j)}\|_\infty$.
7. Prove that $C^k[a,b]$ is a proper dense subspace of $L_p[a,b]$ with $\|.\|_p$ for $1 \leq p < \infty$.
8. There can be many Banach spaces which are completions of a given normed space. But, as far as their linear structure and norm structure are concerned, they are all the same. Find the completions of the following spaces:
   (a) The space $(c_{00}, \|.\|_p)$, for $1 \leq p < \infty$.
   (b) The space $(c_{00}, \|.\|_\infty)$.
   (c) The space $(C(X), \|.\|_p)$, for $1 \leq p < \infty$ and for every measurable subset X of $\mathbb{R}$.
   (d) For $k \in \mathbb{N}$ and $1 \leq p < \infty$, the space $C^k[a,b]$ with respect to the norm $x \mapsto \sum_{j=0}^{k} \|x^{(j)}\|_p$, $x \in C^k[a,b]$. Note that $L_p[a,b]$ can be thought of as the Sobolev space $W^{0,p}[a,b]$ for $1 \leq p < \infty$.
9. Prove that the complement of a subspace L of a normed space X is either dense or empty.
10. If X is a finite dimensional normed space over $\mathbb{R}$ and E is a convex subset of X containing 0, then prove that span$E = X$ iff E° is nonempty.